INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2021

CM2 – Financial Engineering and Loss Reserving
Core Principles
Paper A

Introduction

The Examiners' Report is written by the Chief Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Sarah Hutchinson
Chair of the Board of Examiners
December 2021

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A. General comments on the *aims of this subject and how it is marked*

The aim of Subject CM2 is to develop the necessary skills to construct asset liability models, value financial derivatives and calculate reserves for insurance or guarantees. These skills are also required to communicate with other financial professionals and to critically evaluate modern financial theories.

The marking approach for CM2 is flexible in the sense that different answers to those shown in the solution can earn marks if they are relevant and appropriate. Marks for the methodology are also awarded including marks for using the right method even if an error in an earlier part of the question prevents the final answer from being correct. The marking focusses on rewarding candidates' understanding of the concepts, including their ability to articulate algebra and arguments clearly.

B. Comments on *candidate performance in this diet of the examination.*

This exam was sat online and as a result most questions focussed on applying bookwork and analysing the results. Some of the questions required candidates to apply concepts from the Core Reading to scenarios they might not have seen before and the stronger candidates scored highly here. Average marks were fairly high relative to recent sittings but within the historic norm for the subject.

There was evidence that some candidates found algebra tricky when answering questions in Word. The examiners were lenient with notation when marking these questions, but some candidates missed out on scoring full marks through not explaining their steps. Candidates should note that a pure algebra answer might not always be enough to score full marks.

Candidates should note that rearranging and solving algebra on screen can sometimes be hard if you are used to using pen and paper, so this is a worthwhile skill to practice before the exams.

C. Pass Mark

The Pass Mark for this exam was 63
1,398 presented themselves and 682 passed.

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Solutions for Subject CM2A – September 2021

Q1

(i)

At the end of one year the investment will be worth $100(x + (1 - x) * 1.5) = 150 - 50x$ with probability 0.6 [½]

or $100(x + (1 - x) * 0.5) = 50 + 50x$ with probability 0.4 [½]

Therefore, the expected utility is $0.6 * \ln(150 - 50x) + 0.4 * \ln(50 + 50x)$ [1]

(ii)

Differentiate expected utility: U'(x) = $-50 * \frac{0.6}{150-50x} + 50 * \frac{0.4}{50+50x}$ [1]

Set equal to zero and solve for x:

0.4(150-50x)=0.6(50+50x) [1]

=> x=0.6 [1]

Check that this is a maximum:

U''(x) = - $2500 * \frac{0.6}{(150-50x)^2} - 2500 * \frac{0.4}{(50+50x)^2} < 0$ [1]

Therefore, the investor should invest \$60 in Asset A and \$40 in Asset B [1]

[Total 7]

Part (i) of this question was answered fairly well, but some candidates failed to include the 100 initial wealth in the two scenarios correctly. When calculating expected utility it is important to calculate the total wealth in each scenario before calculating the expected utility of this wealth. Part (ii) was answered well though not all candidates checked the second derivative to score full marks.

Q2

(i)(a)

For the exponential distribution, mean = $\frac{1}{\lambda}$ = 0.5 [1]

For the lognormal distribution, $e^{\mu+\frac{\sigma^2}{2}} = e^{-1.04+\frac{0.833^2}{2}} = 0.500\ (3sf)$ [1]

(b)

For the exponential distribution, var = $\frac{1}{\lambda^2}$ = 0.25

[1]

For the lognormal distribution,

$e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) = e^{-2.08+0.833^2}\left(e^{0.833^2} - 1\right) = 0.250\ (3sf)$ [1]

(c)

For the exponential distribution, the CDF is $F(x) = 1 - e^{-2x}$ [1]

Inverting this and solving, the 99th percentile is at $x = -\frac{1}{2}\log(0.01) = 2.30259$ [1]

For the lognormal distribution:

$P(X < x) = 0.99$

$P(\log(X) < \log(x)) = 0.99$ [½]

$P\left(\frac{logX-(-1.04)}{0.833} < \frac{\log(x)-(-1.04)}{0.833}\right) = 0.99$ [½]

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$\frac{\log(x)+1.04}{0.833} = 2.3263$ [½]
$x = e^{0.833*2.3263-1.04} = 2.45\ (3sf)$ [½]

(ii)
The two distributions have the same mean and variance [½]
But different 99th percentiles [½]
This shows that it is important to consider not just mean and variance [½]
But also tail behaviour of a distribution [½]

(iii)
The lognormal distribution has a higher 99th percentile, which suggests it models the tails of the security's behaviour more appropriately than the exponential distribution [1]
The lognormal distribution has more parameters, so may be more flexible when fitting to historic data [1]
The lognormal distribution in general is a more well-established choice in financial modelling and can lead to useful frameworks like Black-Scholes valuation [1]
The exponential distribution does not have such a framework [1]
Part (i) shows that the lognormal distribution has the heavier upper tail, and this might be a better fit to the heavy upper tail of security prices [1]
Part (i) does not consider the lower tail of the distributions, but the density function for the exponential distribution is largest for the smallest values, which does not fit the observation that security prices cluster around the mean [1]

[Marks available 6, maximum 2]

[Total 12]

Part (i) of this question was answered well by most candidates. Parts (ii) and (iii) tended to be weaker, with many candidates listing generic points about the exponential and lognormal distributions with not enough focus on the specific question asked. These points often scored some marks but could have scored more if tailored to the question.

Q3

For an exponential distribution E(X)=1/α. The mean is 0.5 so α = 2 [1]
R is given by $\lambda M_X(R) = \lambda + cR$ [1]
Using $c = (1+\theta)\lambda m_1$ gives $M_X(R) = 1 + (1+\theta)m_1 R$ [1]
For the exponential distribution $M_X(R) = \frac{2}{2-R}$ therefore [1]
$\frac{2}{2-R} = 1 + 1.1 * 0.5 * R$ [½]
$R(0.1 - 0.55R) = 0$ [½]
$R = 0.1818$ [1]

This question was answered well on the whole, even candidates who reached the wrong final answer did manage to score partial marks for the early steps in the process.

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Q4

(i)

$c_t \leq S_t$ [1]

(ii)

At time T, the call either expires worthless or is exercised [½]
if the call is not exercised, then this is because $K > S_T$. [½]
Then the value of the portfolio is $K > S_T$ [½]
If the call is exercised, then the portfolio value is $S_T - K + K = S_T$ [½]
In either case, the portfolio value is greater than or equal to the share [1]

(iii)

The portfolio is greater than or equal to the value of the share at time T.
By the principle of no arbitrage, it must also be the case that at time t, the portfolio value is greater than or equal to the value of the share: $c_t + Ke^{-(T-t)r} \geq S_t$ [1]

(iv)

An American option is worth at least as much as a European option [½]
Because an American option includes a European in its exercise points [½]
From part (iii), we know that a European option exceeds $S_t - Ke^{(T-t)r}$ [½]
so an American call's value exceeds this too [½]
If this option were exercised, it would pay out $\max(S_t - K, 0)$ [½]
If $S_t > K$, then the exercise value is less than the bound [½]
So the investor would not exercise this option [½]
because it would be better to sell it instead [½]
If $S_t \leq K$, then the option would not be exercised because it is worthless [½]
So in no situation is it optimal for the investor to exercise the option [½]

[Marks available 5, maximum 4]

(v)

The investor may now choose to exercise the option early [1]
The formula in part (iii) does not apply [½]
Because the share accumulates dividends, which was not previously allowed for [½]
Early exercise may also occur if the investor wants to take part in future dividends [½]
This is especially likely to occur if the share is paying out large dividends [½]

[Marks available 3, maximum 2]

[Total 11]

This question was answered well by many candidates. Where marks were lost this tended to be through not explaining steps clearly enough in part (ii) or not giving enough distinct points in part (iv).

Q5

(i)

(10/80)*5% + (50/80)*10% + (20/80)*x = R_M [1]
x – 3% = 2(RM – 3%) => x = 2*RM – 3% [1]
=> 10*5% + 50*10% + 20*(2*RM – 3%) = 80* R_M [1]
=> R_M = (10*5% + 50*10% - 20*3%) / 40 = 12.25% [1]

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OR
(10% - 3%) / (E_M – 3%) = 1 [2]
=> R_M = 10% [2]

Or full credit for any other valid approach.

(ii)
x – 3% = 2*(12.25% - 3%) [½]
=> x = 21.5% [½]

OR
x – 3% = 2*(10% - 3%) [½]
=> x = 17% [½]

Or full credit for any other valid approach.

(iii)
5% - 3% = β*(12.25% - 3%) [½]
=> β = 0.216 [½]

OR
5% – 3% = β *(10% - 3%) [½]
=> β = 0.286 [½]

Or full credit for any other valid approach.

(iv)
There are basic problems in testing the model since, in theory, account has to be taken of the entire investment universe open to investors, not just capital markets [1]
An important asset of most investors, for example, is their human capital (i.e. the value of their future earnings). Models have been developed which allow for decisions over multiple periods and for the optimisation of consumption over time to take account of this [1]
Other versions of the basic CAPM have been produced which allow for taxes, inflation, and also for a situation where there is no riskless asset [1]
In the international situation there is no asset which is riskless for all investors (due to currency risks) so a model has been developed which allows for groups of investors in different countries, each of which considers their domestic currency to be risk-free [1]
There is a discrepancy between the values obtained for the expected return on the market using the security market line and the weightings by market capitalisation [1]
Some other assumptions of CAPM are unrealistic, e.g. everyone has the same estimates for the means/variances/covariances, everyone has the same single time horizon [1]
[Marks available 6, maximum 3]
[Total 9]

Parts (i) to (iii) were answered well by most candidates. There was unfortunately an error in the table provided in the exam paper which meant there was more than one

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possible correct answer to parts (i) through (iii). Credit was given for any valid approach.

Part (iv) tended to be weaker, with many candidates listing generic points about CAPM with not enough focus on the specifics of the question. These points often scored some marks but could have scored more if tailored to the question.

Q6

(i)

$E[(1+i_t)] = 1.04$

$Var[(1+i_t)] = 0.03^2$

Therefore:

$1.04 = \exp(\mu + \frac{\sigma^2}{2})$

$0.03^2 = \exp(2\mu + \sigma^2)\,[\exp(\sigma^2) - 1]$ [1]

So:

$\exp(\sigma^2) - 1 = \frac{0.03^2}{1.04^2}$

$\sigma^2 = \ln\left(\frac{0.03^2}{1.04^2} + 1\right) = 0.00083175$ [1]

And:

$1.04 = \exp(\mu + \frac{0.00083175}{2})$ [1]

$\mu = \ln(1.04) - \frac{0.00083175}{2} = 0.038805$ [1]

(ii)

Working in \$m.

$(1+i_t) \sim Lognormal(\mu, \sigma^2)$

$ln(1+i_t) \sim N(\mu, \sigma^2)$

$ln\left(\prod_{t=1}^{t=n}(1+i_t)\right) = \sum_{t=1}^{t=n} ln(1+i_t)$

[1]

As $\{i_t\}$ are independent:

$\sum_{t=1}^{t=n} ln(1+i_t) \sim N(n\mu, n\sigma^2)$

$\prod_{t=1}^{t=n}(1+i_t) \sim Lognormal(n\mu, n\sigma^2)$

[1]

Let S_n be the accumulated amount after n years of a single investment of one.

$S_n = \prod_{t=1}^{t=n}(1+i_t)$

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So we want the probability that $\$17.5m \times S_3$ is less than \$20m. i.e.

$$P\left(S_3 < \frac{20}{17.5}\right) = P(S_3 < 1.1429) = P(\ln(S_3) < \ln(1.1429))$$

[1]

where $\ln(S_3) \sim N(3\mu, 3\sigma^2) = N(3 \times 0.038805, 3 \times 0.00083175) = N(0.11641, 0.0024953)$ [1]

$$= P\left(N(0,1)) < \frac{\ln(1.1429) - 0.11641}{\sqrt{0.0024953}}\right)$$

$$= P(N(0,1)) < 0.34266) = 0.63407$$

[1]

[Total 9]

This was one of the more difficult questions on the paper, with many candidates slipping up in the algebra for part (i). Part (ii) caused similar problems, and some correct answers were also very brief with not enough workings shown for a five mark question.

Q7

(i)

Suppose that this is not the case: for example, if $e^r \leq d \leq u$ [½]

Then we could borrow S_0 of cash and buy S_0 of stock [½]

At time 0 this would have a net cost of 0 [½]

At time 1 our portfolio would be worth $S_0(d - e^r)$ or $S_0(u - e^r)$, both of which are greater than 0: an example of arbitrage [½]

Similarly if $d \leq u \leq e^r$, then we short S_0 of stock and hold S_0 of cash [½]

This would be worth 0 at time 0 and $S_0(e^r - d)$ or $S_0(e^r - u)$, both of which are greater than 0: another example of arbitrage [½]

(ii)

$q = \frac{e^r - d}{u - d}$ [1]

(iii)

Starting from the relationship in part (i):

$d < e^r < u$ [½]

$0 < e^r - d < u - d$ [½]

$0 < \frac{e^r - d}{u - d} < 1$ [½]

$0 < q < 1$ [½]

[Total 6]

This question was generally answered well. Some candidates lost marks in part (i) by only considering one of the two ways in which the inequality could not hold – it was important to consider $e^r \leq d \leq u$ and $d \leq u \leq e^r$ to score full marks.

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Q8

(i)(a)

$\int_{-\infty}^{\infty}(\mu - x)^2 f_X(x)dx$, where $\mu = E(X)$ [1]

(b)

$\int_{-\infty}^{\mu}(\mu - x)^2 f_X(x)dx$ [1]

(c)

$\int_{-\infty}^{L}(L - x)\, f_X(x)\, dx$ [1]

(ii)

Since different days are independent, we can use the binomial distribution [1]

$P(Number\ of\ Losses \geq 3) = 1 - P(Number\ of\ Losses \leq 2)$ [1]

From the actuarial tables (or by direct calculation), this is 0.001 [1]

(iii)

From part (ii), we know that the chances of this happening are very slim [½]

This suggests that VaR has not been an effective risk measure [½]

Because it has predicted that real-world events were highly unlikely [½]

And because this in practice has cost the trader more money than expected [½]

This may be because the model was not appropriately fitted [½]

e.g. the trader chose an inappropriate approach to fit the distribution to the data [½]

Alternatively, it may be that VaR struggles to appropriately measure tail risk [½]

Many distributions lack sufficiently 'fat tails' to model the extreme behaviour of a market crash [½]

It is possible that the model has only been fitted to historical data and does not include an event similar to the market crash that just occurred [½]

e.g. because the historical data is 'milder' than the event that has just occurred [½]

Alternatively, it may be that other assumptions are inappropriate [½]

e.g. the assumption that different days are independent [½]

e.g. using a one-day VaR did not cover a long-enough timeframe to capture the security's risk and a one-week/one-month VaR may have been better [½]

The exact model used by the trader might be overly complex [½]

There might be a problem with the software tools used to run the VaR model [½]

VaR doesn't give a measure of how bad things could get if the level L is breached – this doesn't make it a very effective risk measure [½]

[Marks available 8, maximum 4]

[Total 10]

Most candidates answered part (i) well here. Part (ii) tended to be weaker with some candidates calculating 0.1^3 = 0.001, which appears to give the right answer but is not a valid calculation. Part (iii) was answered fairly well, though many candidates did not make enough distinct and valid points for full marks.

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Q9

(i)

$E[i_t] = j = \frac{1}{2}(i_1 + i_2)$ [½]

$Var[i_t] = s^2 = E\left[{i_t}^2\right] - E[i_t]^2$ [½]

$= \frac{1}{2}\left({i_1}^2 + {i_2}^2\right) - (\frac{1}{2}(i_1 + i_2))^2$ [½]

$= \frac{1}{2}\left({i_1}^2 + {i_2}^2\right) - \frac{1}{4}\left({i_1}^2 + {i_2}^2 + 2i_1 i_2\right)$ [½]

$= \frac{1}{4}\left({i_1}^2 + {i_2}^2\right) - \frac{1}{2}i_1 i_2$ [½]

$= (\frac{1}{2}(i_1 - i_2))^2$ [½]

(ii)

$E[S_{20}] = (1 + j)^{20} = 4.5/2 = 2.25$ [½]

Therefore $j = \sqrt[20]{2.25} - 1 = 0.0413797$ [1]

$Var[S_{20}] = (1 + 2j + j^2 + s^2)^{20} - (1 + j)^{40} = (0.75/2)^2 = 0.375^2$ [½]

$s^2 = \sqrt[20]{0.375^2 + 1.0413797^{40}} - 1 - 2 \times 0.0413797 - 0.0413797^2 = 0.0014867$ [½]

$s^2 = 0.0014867 = (\frac{1}{2}(i_1 - i_2))^2$ [1]

$i_1 > i_2$ therefore take the positive root:

$i_1 - i_2 = 2 \times \sqrt{0.0014867} = 0.077115$ [½]

$i_1 + i_2 = 2 \times j = 2 \times 0.0413797 = 0.082759$ [½]

$2i_1 = 0.077115 + 0.082759 = 0.159874$ [½]

$i_1 = 0.079937$ [½]

$i_2 = i_1 - 0.077115 = 0.002822$ [½]

[Total 9]

This was the most difficult question on the paper. Most candidates scored some marks for the initial algebra, but only the better prepared candidates managed to work through to the correct final answers.

Q10

(i)

We can use N(t), the claim number process at time t. For a fixed value of t, if no claims have occurred by time t, $T_1 > t$, hence:

$P(T_1 > t) = P(N(t) = 0) = \exp(-\lambda t)$ [1]

Therefore $P(T_1 \leq t) = 1 - \exp(-\lambda t)$ [½]

Which is the CDF of an exponential distribution, therefore T_1 has an exponential distribution with parameter λ [½]

(ii)

For i = 2, 3, … let the random variable T_i denote the time between the *(i-1)*-th and the *i*-th claims. Then:

$P(T_{n+1} > t \mid \sum_{i=1}^{n} T_i = r) = P(\sum_{i=1}^{n+1} T_i > T + r \mid \sum_{i=1}^{n} T_i = r)$ [½]

$= P(N(t + r) = n \mid N(r) = n)$

$= P(N(t + r) - N(r) = 0 \mid N(r) = n)$ [½]

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Because we are working with a Poisson process, when $s < t$, the number of claims in the time interval $(s, t]$ is independent of the number of claims up to time s. [½]
Therefore $= P(\, N(t+r) - N(r) = 0 \mid N(r) = n) = \; P(\, N(t+r) - N(r) = 0)$ [½]

Finally:
$P(\, N(t+r) - N(r) = 0) = P(\, N(t) = 0) = exp\{-\lambda t\}$ [½]
since the number of claims in a time interval of length r does not depend on when that time interval starts. Thus, inter-event times also have an exponential distribution with parameter λ [½]

(iii)(a)
In general, $\Psi(U)$ is a non-increasing function of U [½]
Taking the derivative with respect to U of the result in the question gives:
$\frac{d}{dU}\Psi(U) = \frac{-\theta}{1+\theta}\Psi(U)$ [1]
Which is negative since θ>0 (or just by inspection) [½]
Hence $\Psi(U)$ is a decreasing function of U [½]
This is a more precise statement than the general case [½]
It is intuitively sensible that $\Psi(U)$ should be a decreasing function of U [½]
An increase in U represents an increase in the insurer's surplus without any corresponding increase in claim amounts. Thus, an increase in U represents an increase in the insurer's security and hence will reduce the probability of ruin [1]
[Marks available 4½, maximum 2½]

(b)
In general, $\Psi(U)$ is a non-increasing function of θ [½]
Taking the derivative with respect to θ of the result in the question gives:
$\frac{d}{d\theta}\Psi(U) = -(1+\theta)^{-1}\Psi(U) - U(1+\theta)^{-2}\Psi(U)$ [1]
Which is negative since θ, U and $\Psi(U)$ are all positive quantities [½]
Alternatively, rearranging the RHS to give Psi(U) = 1/(1+theta) * exp(U*(1/(1+theta)-1)) shows that both 1/(1+theta) and exp(U*(1/(1+theta)-1)) decrease with theta [1]
Therefore the product must also decrease, without the need to differentiate [½]
Hence $\Psi(U)$ is a decreasing function of θ [½]
Logically this makes sense as increasing θ increases the rate of premium income for the insurer [½]
So the insurer would be more able to withstand claims, hence the probability of ruin would be lower [½]
[Marks available 5, maximum 2½]
[Total 10]

This question was answered well by most candidates. Part (iii) caused the most difficulty, with some candidates just stating the relationships which was not enough to score strongly in a question that requires explanation and logical reasoning.

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Q11

(i)

For the utility function to respect the lemurs being risk-averse and non-satiated, it must have the following properties:

$U'(w) > 0$ [½]

$U''(w) < 0$ [½]

Going through each function we find the following:

(a)

$U'(w) = 1 + 2w$ [½]

This is smaller than 0 when $w < -0.5$ [1]

$U''(w) = 2$ [½]

Hence $U''(w) > 0$ [1]

This cannot be a valid utility function [1]

according to either measure [½]

[Marks available 4½, maximum 2½]

(b)

$U'(w) = \frac{\gamma w^{\gamma-1}}{\gamma} = w^{\gamma-1}$ [½]

This is positive [½]

because $w > 0$ [½]

$U''(w) = (\gamma - 1)w^{\gamma-2}$ [½]

This is negative [½]

because we are told that $\gamma < 1$ [½]

So this could be a valid function [½]

[Marks available 3½, maximum 2½]

(c)

$U'(w) = 1 - 4w$ [½]

$U''(w) = -4 < 0$ [½]

The second derivative is acceptable [½]

but the first is not acceptable [½]

because it is not valid at e.g $w = 1$, where the first derivative is -3 (or any other counterexample using $\frac{1}{4} \leq w \leq 3$) [½]

So this cannot be a valid utility function [½]

[Marks available 3, maximum 2]

(ii)

Scenario A: expected utility is $E\big(U(w)\big) = 0.5(\ln(4) + \ln(1))$ [½]

$E\big(U(w)\big) = \ln(2)$ [½]

Scenario B: expected utility is $E\big(U(w)\big) = \ln(2.1)$ [½]

The expected utility to the lemurs is larger under scenario B [½]

So if the utility function is appropriate, they should prefer scenario B [½]

This is consistent with the experiment's findings [½]

[Total 11]

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This question was answered well apart from some slip-ups with differentiating utility functions. In parts (i)(a) and (i)(c) candidates only had to give one proof that the function was invalid to score full marks in that part. In part (ii) some candidates calculated U'(w) and U''(w) which does not answer the question about the findings of the experiment.

[Paper Total 100]

END OF EXAMINERS' REPORT

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