Time series modelling of Gompertz-Makeham mortality curves: historical analysis, forecasting and life insurance applications

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Abstract

The Continuous Mortality Investigation (CMI) of the Institute and Faculty of Actuaries has for a number of years based its graduated tables of assured life, annuitant and pensioner mortality on Gompertz-Makeham formulae. In this thesis, we consider two-dimensional data sets consisting of the number of deaths and the exposed to risk at a range of ages in a range of calendar years. Having fitted a Gompertz-Makeham model to the data for each calendar year, we fit univariate time series models to represent the behaviour over time of the Gompertz-Makeham parameters. Cohort effects are allowed for by applying a multiplicative factor depending on year of birth to the fitted force of mortality. Prediction intervals for the future parameters of the model are calculated. Sample values of immediate and deferred annuities are presented, based on stochastic mortality simulations and a deterministic interest rate. An application to risk-based capital calculations, under the Individual Capital Assessment (ICA) regime of the Financial Services Authority (FSA), is presented.

Key words

Mortality; Gompertz-Makeham Models; Time Series; Cohort Effects; Prediction Intervals; Annuities; Individual Capital Assessment

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1: Introduction

It is necessary to model future mortality in order to assess the level of reserves and capital required for a portfolio of immediate or deferred annuities held by a life insurance company or by a pension fund, or for a portfolio of assurance contracts held by a life insurance company. Such modelling is particularly important for life insurance companies with significant guaranteed annuity option liabilities, as improvements in longevity have the potential to bring about much larger percentage increases in the value of these options than in the value of the underlying annuities. It is also necessary to model future mortality to price annuity and life insurance contracts, and to value securities whose payoffs are contingent on future mortality.

Figure 1.2 shows historical plots of the force of mortality as a function of age and calendar year. The data sets these plots relate to will be described in the 'Data' section below. We can see in all cases that the bands of colour in the plots slope upwards, indicating a persistent improvement in longevity over the period of the data. This is a key feature of the data that needs to be captured in a mortality model, otherwise the reserves and prices calculated by the model for annuity contracts cannot be expected to be adequate and those for life insurance contracts may be excessive. In order to capture this feature, the model needs to consider mortality as a two-dimensional function of calendar year as well as of age.

The rate of improvement in mortality shown in the plots in Figure 1.2 is not uniform but varies significantly by age, by calendar year and between the three data sets. Given the high values shown in Figure 1.1 for the exposed to risk for each of the data sets, it seems unlikely that these variations result solely from random fluctuations. As a result of the wide range of factors that have been observed to affect mortality, including standards of health care and lifestyle factors such as diet and smoking, it is unlikely that we will be able to obtain a definitive explanation for all these variations. It is still less likely that we will be able to make a reliable deterministic forecast of how these underlying factors will evolve in the future and hence derive a reliable deterministic mortality projection. We shall therefore develop a stochastic mortality model, explicitly recognising the uncertainty of future mortality. This is particularly important in reserving and capital assessment applications, as an institution with liabilities dependent on future mortality needs to consider this uncertainty to ensure that it has sufficient capital to meet the liabilities with a high level of confidence. Furthermore, there may be a regulatory requirement on the institution to hold sufficient capital to meet its liabilities with a defined level of confidence, and we shall consider an example of this in Section 5.6.

A number of stochastic mortality models considering mortality as a two-dimensional function of both age and calendar year have been fitted in previously published papers and some of the main examples are discussed below.

P-spline model

Currie *et al*. (2004) propose a non-parametric model, the *P-spline model*, of the form:

$$\log \mu_{xt} = \sum_i \sum_j \theta_{ij} B_i^a(x) B_j^y(t),$$

where μ_{xt} is the force of mortality at age x in calendar year t, the θ_{ij} are parameters to be estimated and the functions B_i^a and B_j^y are cubic basis splines, as defined in, for example, de Boor (2001). A penalty is imposed on parameter estimates $\{\theta_{ij}\}$ which do not vary smoothly either with i and j (an *age-period penalty*) or with i and $j - i$ (an *age-cohort penalty*). The level of this penalty determines the balance between smoothness and goodness of fit. The estimation of the θ_{ij} is then by maximising the log-likelihood less the penalty.

This model is considered by the CMI in CMI (2005), and further in CMI (2006) where it is fitted to the CMI's male assured lives data set and to England and Wales population data for both males and females. Being non-parametric, it has the advantage of flexibility, in that it can fit any pattern of mortality as a function of age and of calendar year that is in evidence in the data. However, the model has been criticised for producing projections of future mortality that

are unduly sensitive to the last year in the data set. In addition, as the model does not impose any structure on the dependence of mortality on age, there can be no certainty that the projected curves of the logarithm of mortality as a function of age (hereafter simply referred to simply as *mortality curves*) in future years will be reasonable. Finally, the procedure for fitting the model only produces percentiles of the distribution of mortality rates at each age in each future year. For stochastic modelling, we are more likely to be interested in generating a number of sample paths containing projected mortality rates at all ages in all future years.

Lee-Carter model

Lee and Carter (1992) propose the following model:

$$\log \mu_{xt} = \alpha_x + \beta_x \kappa_t ,$$

where the α_x, β_x and κ_t are parameters to be estimated. This model is considered by the CMI in CMI (2005), and further in CMI (2007) where it is fitted to the CMI's male assured lives data set and to England and Wales population data for both males and females. A time series model is then fitted to the kappa parameters, and projected into the future to calculate future mortality rates and corresponding sample annuity values.

A key advantage of this model over the P-spline model is that, as future projections can be calculated simply by projecting the time series of kappa parameters, both sample paths and percentiles of the distribution of future mortality rates can readily be calculated. Another advantage of this model compared with the P-spline model is that the parameters have a clear interpretation – the alpha parameters represent the variation of mortality with age, the kappa parameters represent the improvement of mortality over time and the beta parameters provide for the possibility that mortality may improve more rapidly at some ages than at others. However, CMI (2007) concludes that this model does not fit United Kingdom data well because it cannot incorporate *cohort effects*. This refers to the fact that certain generations have consistently exhibited either particularly high or particularly low mortality improvements compared with the previous generation. For example, Willets *et al*. (2004) identify the generation centred on year of birth 1931 for England and Wales data for both males and females, and the generation centred on year of birth 1926 for CMI data, as exhibiting particularly high mortality improvements compared with the previous generation. Willets (2004) suggests that this phenomenon is mainly caused by a decline in smoking prevalence and by changes in diet in early life.

Lee and Carter (1992) estimated the parameters of their model by minimising the following sum of squares:

$$\sum_x \sum_t \left(\log \hat{\mu}_{xt} - \alpha_x - \beta_x \kappa_t\right)^2 ,$$

where $\hat{\mu}_{xt} = \dfrac{D_{xt}}{E_{xt}}$ and D_{xt} and E_{xt} are, respectively, the number of deaths and the central exposed to risk in the data at age x in calendar year t. They then adjusted the kappa parameters so as to match the actual and expected total numbers of deaths in each calendar year. However, Brouhns *et al*. (2002) criticise this procedure for placing undue weight on regions of the data set where there are few deaths and where the standard errors of the force of mortality are therefore large. Brouhns *et al*. instead model the number of deaths at age x in calendar year t as a Poisson random variable with parameter equal to E_{xt} multiplied by the Lee-Carter force of mortality, *i.e.*

$$E_{xt} \exp(\alpha_x + \beta_x \kappa_t) .$$

The alpha, beta and kappa parameters are then estimated by maximum likelihood.

Lee and Carter (1992), Brouhns *et al*. (2002) and CMI (2007) all treat the alpha, beta and kappa parameters as fixed quantities estimated from the data. The stochastic variation in future projections then comes entirely from the innovation terms in the time series process fitted to the kappa parameters. However, this does not allow for the fact that the parameter estimates are subject to uncertainty. It also does not allow for the fact that the new data which emerge over time will in practice lead us to update our estimates of the alpha and beta parameters, and of the parameters of the time series process governing the kappa parameters. To address these issues, Czado *et al*. (2005) develop a Bayesian approach. Together with an assumed prior distribution, the data are used to derive a joint posterior distribution which is updated as new data emerge and gives interval estimates of the parameters. Simulation from this posterior distribution is a non-trivial task and the authors use the technique of *Markov chain Monte Carlo* to construct a Markov chain whose stationary distribution is the required posterior distribution.

Extensions to the Lee-Carter model incorporating cohort effects

Renshaw and Haberman (2006) propose an extension to the Lee-Carter model incorporating an explicit allowance for cohort effects:

$$\log \mu_{xt} = \alpha_x + \beta_x^{(1)} \kappa_t + \beta_x^{(2)} \gamma_{t-x}.$$

The authors fit this model to England and Wales population data, for both males and females. This addresses the criticism of the Lee-Carter model as not allowing for cohort effects. However, CMI (2007), which includes a limited investigation of this model, reports some problems with its convergence and robustness. The iterative procedure used by Renshaw and Haberman to estimate the parameters was found to arrive at different solutions according to the parameter values that were taken as the starting point for the iterations. For certain subsets of CMI male assured lives data and of England and Wales male data, the model completely failed to converge. One possible reason for these problems is that convergence is extremely slow as a result of the likelihood function being relatively flat in some directions, because it is relatively difficult to distinguish between a cohort effect and a combination of age and period effects, and much steeper in others. Thus the iterations may simply not have been allowed to run for long enough. However, Cairns *et al*. (2008) provide some evidence to suggest that the problems with the model are more fundamental than this, in that the likelihood function has multiple maxima. Cairns *et al*. successively added additional calendar years to the data and found that in some cases, adding an additional year's data caused the parameter estimates to jump to a solution with qualitatively different behaviour. Cairns *et al*. also identify issues with the plausibility of future projections produced by the model in that there is more uncertainty in the projections at ages around 65 than at the oldest ages in the data, which does not seem reasonable given that there is more data at ages around 65.

Cairns *et al*. (2008) consider simplifying the Renshaw-Haberman model by making the beta parameters independent of age, leading to the so-called *age-period-cohort model*, which the authors refer to as model M3 whereas the Renshaw-Haberman model is referred to as model M2. The authors find that making the beta parameters independent of age resolves the issues with the convergence and robustness of the Renshaw-Haberman model. However, under this simplified model, any period effect and any cohort effect must have the same percentage impact on the force of mortality at all ages, and this might be considered too limiting.

Cairns-Blake-Dowd (CBD) model

Cairns *et al*. (2006) propose the following model:

$$q_{xt} = \frac{\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x\right)}{1 + \exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x\right)}.$$

Note that this model is fitted to initial mortality rates q_{xt} rather than to forces of mortality μ_{xt}. The authors fit it to England and Wales male data for ages 60 and above. This model is more

flexible than the Lee-Carter model in terms of the ways in which mortality can evolve over time, with trends in the $\kappa^{(0)}$ parameters having proportionately more impact on mortality at the younger ages in the data set and trends in the $\kappa^{(1)}$ parameters having more impact at the older ages. The model is considerably less flexible than the Lee-Carter model in terms of the ways in which mortality can vary with age. The authors find no evidence that the model does not allow a sufficiently wide range of shapes of the mortality curve as a function of age for the data set they consider, but this would not necessarily be the case for other data sets, particularly those extending to ages below 60. The authors find that the shapes of the $\kappa^{(0)}$ and $\kappa^{(1)}$ parameter graphs they obtain implicitly reflect cohort effects, and this motivates the development of extensions of the CBD model that incorporate an explicit allowance for cohort effects.

Extensions to the CBD model incorporating cohort effects

Cairns *et al*. (2007) propose three different extensions to the CBD model incorporating cohort effects explicitly, referring to the CBD model as model M5 and to the extensions as models M6, M7 and M8. The extensions are as follows:

$$\text{M6: } q_{xt} = \frac{\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x + \gamma_{t-x}\right)}{1+\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x + \gamma_{t-x}\right)},$$

$$\text{M7: } q_{xt} = \frac{\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x + \kappa_t^{(2)} x^2 + \gamma_{t-x}\right)}{1+\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x + \kappa_t^{(2)} x^2 + \gamma_{t-x}\right)},$$

$$\text{M8: } q_{xt} = \frac{\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x + \gamma_{t-x}\left(x_c - x\right)\right)}{1+\exp\left(\kappa_t^{(0)} + \kappa_t^{(1)} x + \gamma_{t-x}\left(x_c - x\right)\right)},$$

where x_c is a parameter to be determined. The authors find that all three of these models achieve a very significant improvement over M5 in goodness of fit to both England and Wales and United States male population data, with M7 giving a sufficient improvement over M6 to justify the introduction of the $\kappa^{(2)}$ parameters. The criterion used to measure goodness of fit here is the Bayes Information Criterion (BIC), which is discussed further in Section 2.5 of this paper. The authors find that models M7 and M6 (which is a special case of M7) are adequately robust but raise concerns about the robustness of model M8 in the context of US data. This is developed further in Cairns *et al*. (2008) where it is found that model M8 fails to produce plausible projections of future mortality at ages over 65 for US data.

Gompertz-Makeham (GM) models

The graduated one-dimensional tables of assured life, annuitant and pensioner mortality produced by the CMI extend to ages below 60, with the annuitant and pensioner tables including early as well as normal retirements. In most cases, it is found that a model similar to the CBD model cannot adequately capture the age structure of the data. As a result, for a number of years, the CMI have based most of their graduations on *Gompertz-Makeham models of order* (*r*,*s*), or *GM*(*r*,*s*) models, for various non-negative integers *r* and positive integers *s*. This family of models was proposed by Forfar *et al.* (1988). The GM(*r*,*s*) model is defined by:

$$\mu_x = \sum_{i=0}^{r-1} \kappa^{(i)} x^i + \exp\left(\sum_{j=0}^{s-1} \kappa^{(r+j)} x^j\right),$$

where μ_x is the force of mortality at age *x* and $\kappa^{(0)},\ldots,\kappa^{(r+s-1)}$ are parameters to be estimated. In the case where *r* = 0, we define the empty sum $\sum_{i=0}^{r-1} \kappa^{(i)} x^i$ to have the value 0. Typically

mortality at younger ages is driven primarily by the parameters $\kappa^{(0)},\ldots,\kappa^{(r-1)}$ and mortality at older ages is driven primarily by $\kappa^{(r)},\ldots,\kappa^{(r+s-1)}$.

Figure 1.3 shows fitted mortality curves under several different GM models. The data set considered is the CMI data set, which will be described in the 'Data' section below, and the calendar year considered is 2000. The changes for the GM(1,3) model compared with the GM(1,2) model, and for the GM(2,3) model compared with the GM(1,3) model, have been exaggerated to emphasise the features that have changed. We see from Figure 1.3(i) that under the GM(0,2) model, the logarithm of the force of mortality is a linear function of age. It is found that this understates mortality at the youngest ages of the data set, and Figure 1.3(ii) shows that the main effect of introducing $\kappa^{(0)}$ parameters is to rectify this by making mortality increase more slowly than exponentially with age at the youngest ages of the data set. Similarly, Figure 1.3(iii) shows that the main effect of introducing $\kappa^{(5)}$ parameters is to make mortality increase more slowly than exponentially with age at the oldest ages of the data set. Figure 1.3(iv) shows that under the GM(2,3) model, the graph of the logarithm of mortality is broadly linear below age 45 (with a lower slope than at higher ages), rather than for the slope of the graph to continue to decrease as the age approaches the youngest age of the data set (30).

In this paper, we shall extend the GM family of models to the two-dimensional case by making the kappa parameters functions of time. Thus we shall consider models of the form:

$$\mu_{xt} = \sum_{i=0}^{r-1} \kappa_t^{(i)} x^i + \exp\left(\sum_{j=0}^{s-1} \kappa_t^{(r+j)} x^j \right),$$

where μ_{xt} is the force of mortality at age x in calendar year t. We shall fit models of this form to the three data sets described below.

Data

The majority of the papers referred to above consider only population data. We, however, will also consider CMI male data, which can be expected to be more relevant to a typical life insurance company's liabilities. The population data we shall consider will be England and Wales data for both males and females.

The CMI male data we shall use represent the mortality experience of male assured lives holding endowment or whole life assurance policies with UK life insurance companies that contributed to the CMI's investigation over the period concerned. The data consist of the number of deaths and the central exposed to risk at each of the ages 30-90 nearest birthday in each of the calendar years 1947-2005, a total of 3,599 data cells. We shall not consider CMI data for females, or for pensioner or annuitant data sets, here as these data sets have either insufficient volume or an insufficiently long history to draw reliable conclusions.

The England and Wales data, for both males and females, were taken from the Human Mortality Database (www.mortality.org) maintained by the University of California, Berkeley (USA) and by the Max Planck Institute for Demographic Research (Germany). The data were originally provided by the Office for National Statistics (ONS). The data for each gender consist of the number of deaths and the central exposed to risk at each of the ages 30-89 last birthday in each of the calendar years 1962-2005, a total of 2,640 data cells. The data were downloaded on 20 March 2008.

In what follows, we shall denote the number of deaths in the data at age x in calendar year t by D_{xt}, where x is defined as the age nearest birthday in the case of CMI data and as the age last birthday in the case of England and Wales data. We shall denote the central exposed to risk at age x in calendar year t by E_{xt}.

Figure 1.1 shows the logarithm of the exposed to risk at each age in each calendar year for each data set. Regions of these graphs where the exposed to risk is high are coloured red

and those where the exposed to risk is low are coloured blue. We can make the following observations from Figure 1.1:

- Noting the different scales of the three graphs, the CMI data is a smaller data set than either of the England and Wales data sets. This is because it relates only to assured lives rather than to the general population.

- The exposed to risk in all three graphs decreases towards the top of the age range of the data. For England and Wales data, this can be explained by relatively few lives surviving to these high ages. For CMI data, another factor is that relatively few retired lives would be expected to hold life insurance policies, and this results in the significant decrease in exposed to risk starting at a rather younger age in CMI data than in the England and Wales data sets.

- In all three graphs, relatively low values of the exposed to risk can be seen at years of birth in the second half of the 1910s. This is likely to be a result of low birth rates during the First World War. For the male data sets, the fact that a substantial proportion of these lives were killed in the Second World War is also likely to be a significant factor.

- The England and Wales data sets show relatively high values of the exposed to risk for lives born in the years immediately following the Second World War and in the 1960s. This can be identified with high birth rates during those periods.

- The CMI data set shows a significant decrease in the exposed to risk in the last few years of the data at ages below a typical retirement age. The decrease in exposed to risk begins rather earlier at the younger ages, where the policies tend to be relatively new. This can be explained by the tendency in recent years for individuals to take out repayment mortgages, under which a life insurance company provides a term assurance policy, rather than endowment mortgages, under which the insurer provides an endowment. The latter policies are included in the data set but the former are not. The decline in popularity of endowments can largely be explained by the ending of tax relief on premiums under this business for policies taken out from 1984.

Figure 1.2 shows the logarithm of the crude force of mortality, $\log\left(\frac{D_{xt}}{E_{xt}}\right)$, at each age in each calendar year for each data set, with a red colour representing high mortality and a blue colour representing low mortality.

It is clear from all the graphs in Figure 1.2 that there is an increasing trend of mortality with age. In addition, as mentioned in the opening remarks, the bands of colour in the graphs slope from bottom left to top right, indicating a trend for mortality to improve over time. There is evidence that the rate of this improvement has been highest for CMI data and lowest for England and Wales female data.

Figure 1.4 shows graphs of the logarithm of the crude force of mortality for CMI data as a function of age x for t = 1960, 1975, 1990 and 2005 respectively. Figure 1.5 shows the same information for England and Wales male data and Figure 1.6 shows the same information for England and Wales female data, for t = 1970, 1980, 1990 and 2000 respectively. 95% confidence limits for the force of mortality are also shown, *i.e.* the lower limit is the 2.5th percentile and the upper limit is the 97.5th percentile. These confidence limits assume that the number of deaths at each age in each calendar year has a Poisson distribution. The graph for t = 2005 for CMI data (Figure 1.4(iv)) uses an age range of 35-90 instead of 30-90 because of the paucity of data at the youngest ages.

The following observations can be made from Figures 1.4-1.6:

- As for Figure 1.2, all the figures clearly demonstrate an increasing trend of mortality with age. Some evidence can be seen of the increasing trend being smaller in percentage terms at younger ages, *i.e.* the curves are less steep at younger ages.

- As in Figure 1.2, some tendency can be seen for mortality at each age to decrease over time.

- Figure 1.4 exhibits greater volatility and wider confidence intervals than Figures 1.5 and 1.6. This is because the CMI data is a smaller data set than the England and Wales data sets.

- In England and Wales data, for a given age and calendar year, the force of mortality for females is lower than that for males.

- In both CMI data and England and Wales data, the confidence intervals widen towards the bottom of the age range. This is because there are relatively few deaths at these ages.

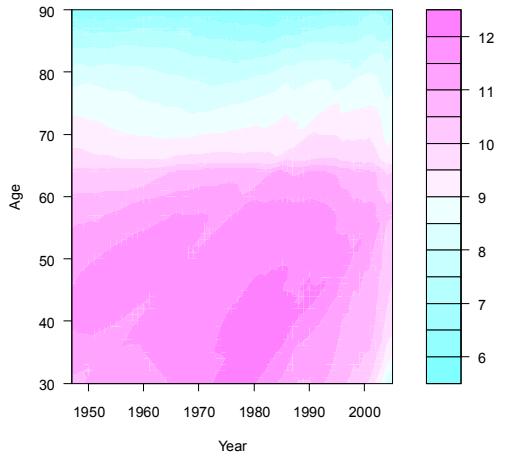
- In CMI data, some widening of the confidence intervals is visible towards the top of the age range. Although mortality is high at these ages, the exposed to risk is much lower than around the middle of the age range.

Structure of the thesis

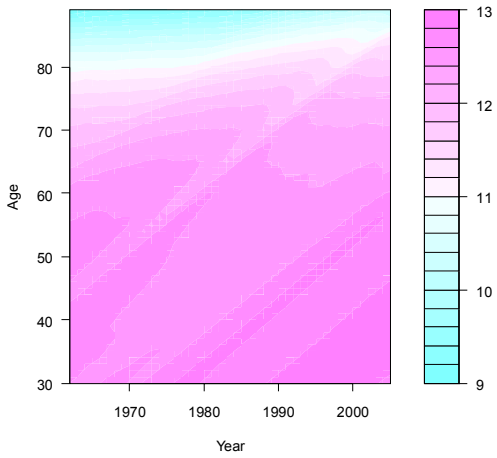
Chapter 2 fits GM(r,s) models, for various values of r and s, to the three data sets, and arrives at a conclusion as to the most appropriate values of r and s on which to base future mortality projections. Chapter 3 considers a simple method of adjusting for cohort effects, introducing a further time series of parameters indexed by year of birth rather than by calendar year. Chapter 4 fits univariate time series models to the parameter estimates calculated in Chapters 2 and 3. Chapter 5 presents stochastic projections of the future values of the parameters and resulting sample immediate and deferred annuity functions, calculated at a deterministic interest rate. An application to risk-based capital calculations, under the Individual Capital Assessment (ICA) regime of the Financial Services Authority (FSA), is given. This chapter contains more applications and gives more discussion of the observations than can be found in most previously published papers on stochastic mortality models. Chapter 6 gives our conclusions.

Figure 1.1 – Logarithms of the exposed to risk for the three data sets **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

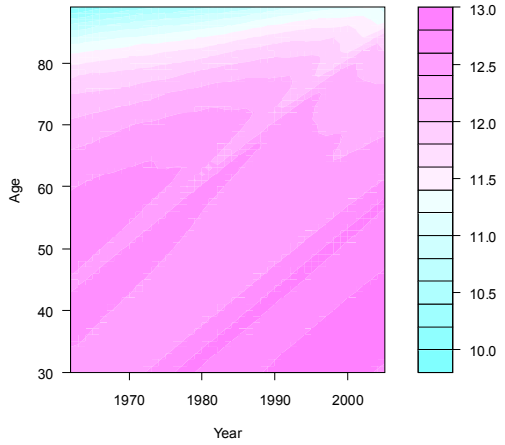
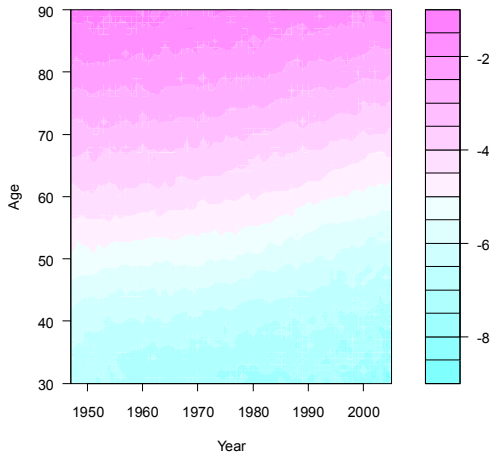


Figure 1.2 – Logarithms of the crude force of mortality for the three data sets **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



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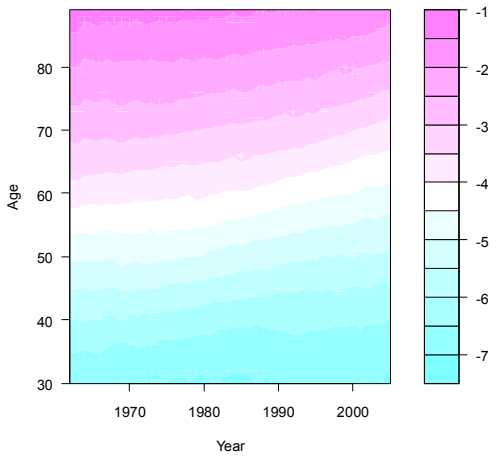

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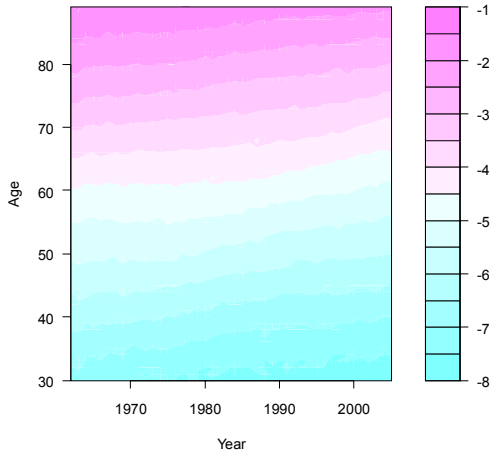
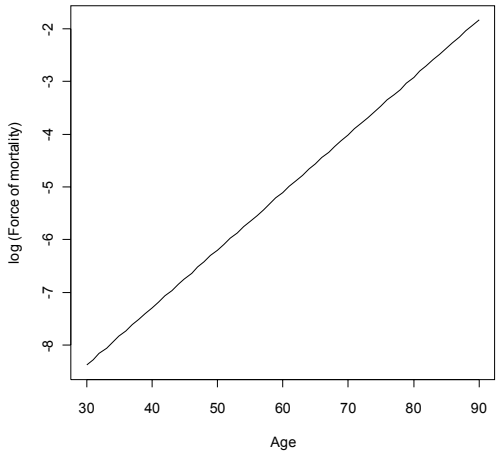


Figure 1.3 – Fitted mortality curves under various GM models, with exaggeration to highlight certain features **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

(i)



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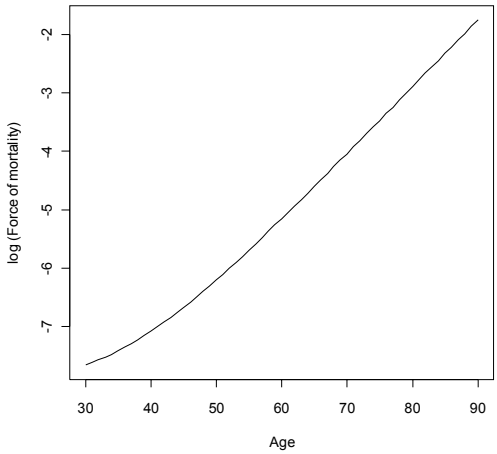

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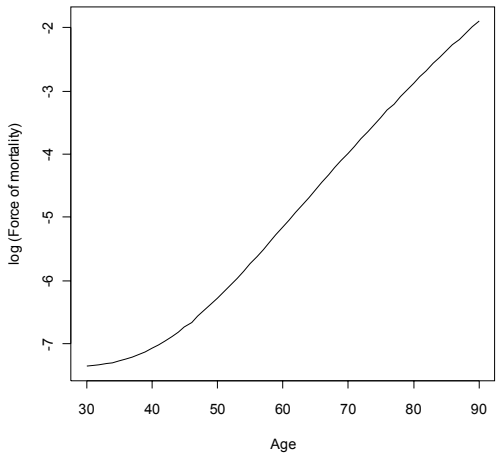

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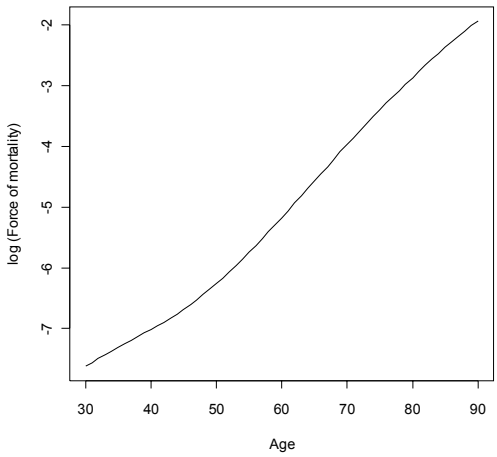
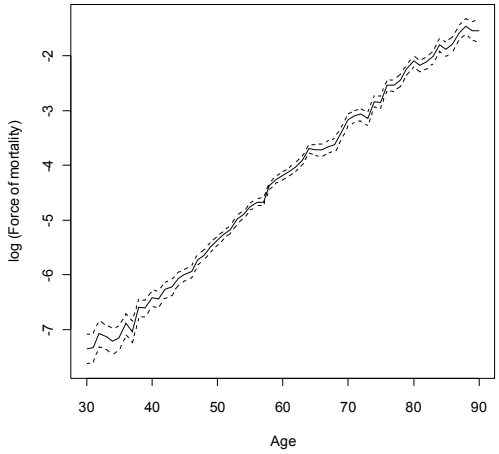


Figure 1.4 – CMI crude force of mortality as a function of age for various calendar years **–** solid curve = central estimate, dashed curves = 95% confidence limits (2.5^{th} and 97.5^{th} percentiles) **–** (i) 1960, (ii) 1975, (iii) 1990, (iv) 2005

(i)



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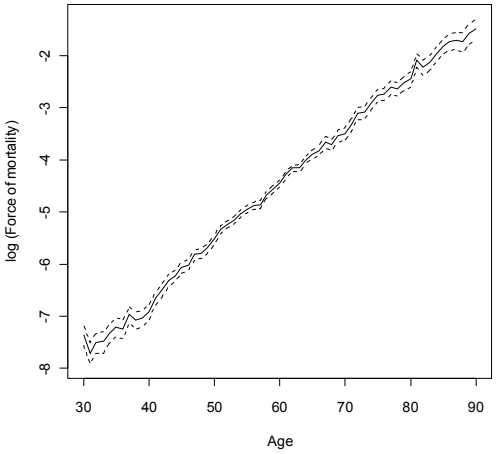
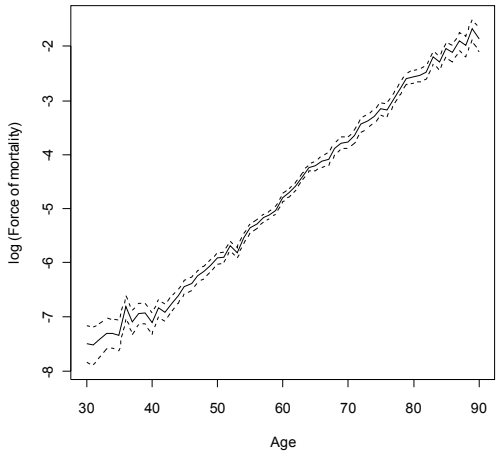


(iii)



(iv)

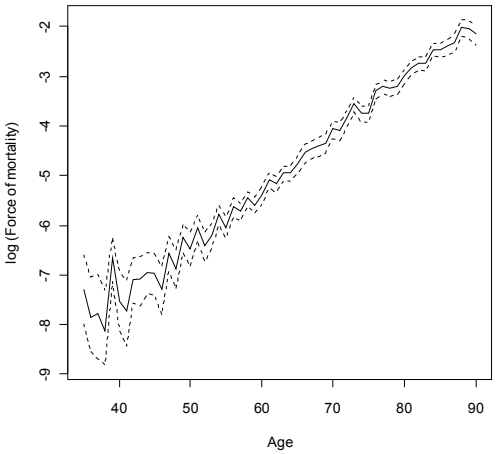
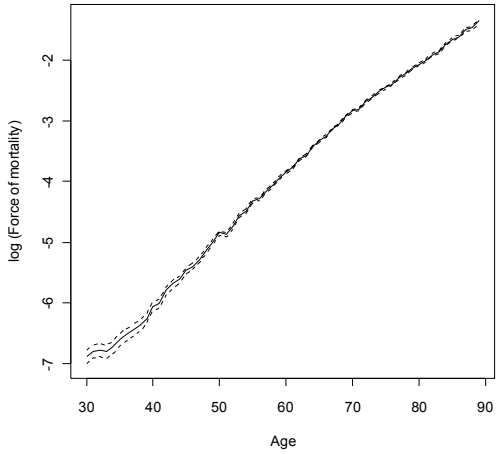
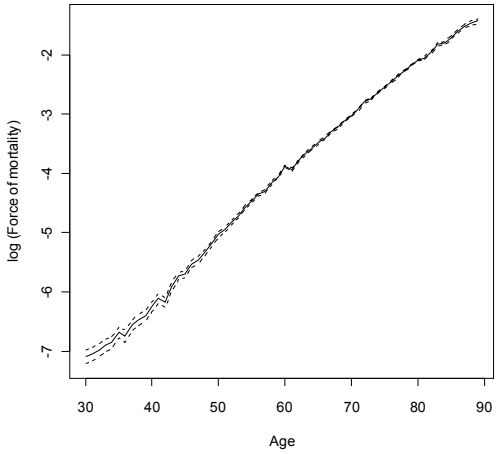


Figure 1.5 – England and Wales male crude force of mortality as a function of age for various calendar years **–** solid curve = central estimate, dashed curves = 95% confidence limits (2.5th and 97.5th percentiles) **–** (i) 1970, (ii) 1980, (iii) 1990, (iv) 2000

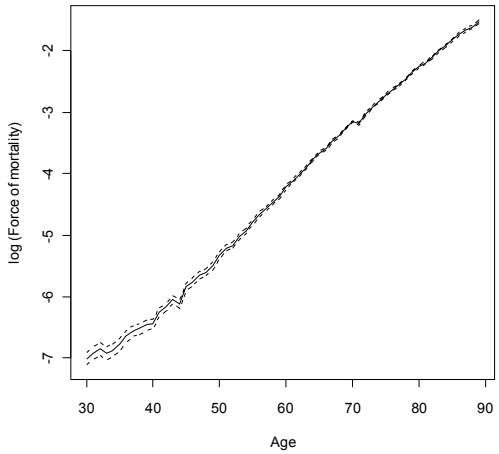
(i)



(ii)



(iii)



(iv)

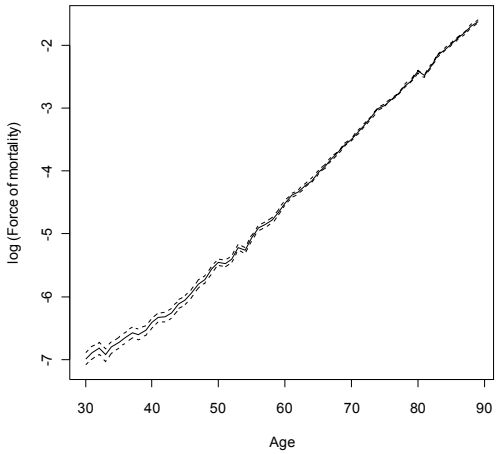
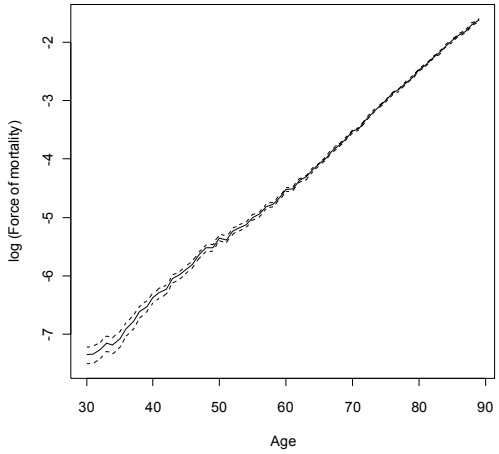
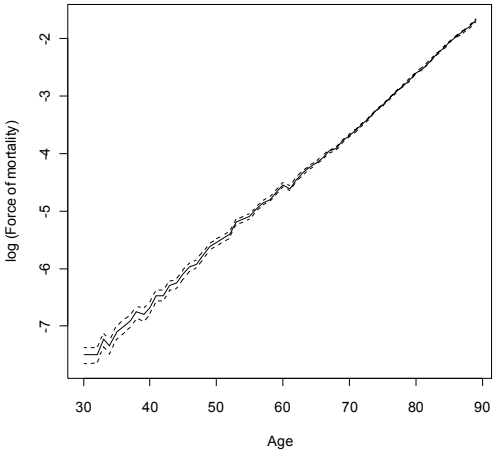


Figure 1.6 – England and Wales female crude force of mortality as a function of age for various calendar years **–** solid curve = central estimate, dashed curves = 95% confidence limits (2.5^{th} and 97.5^{th} percentiles) **–** (i) 1970, (ii) 1980, (iii) 1990, (iv) 2000

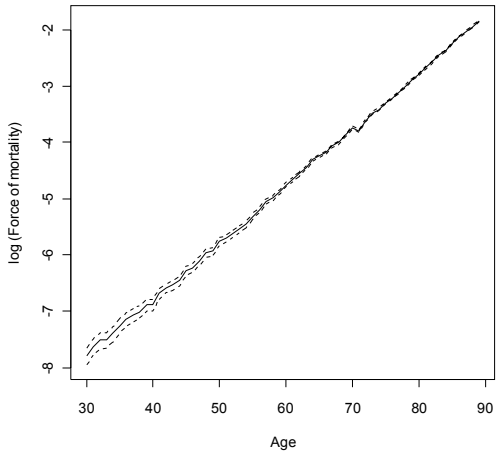
(i)



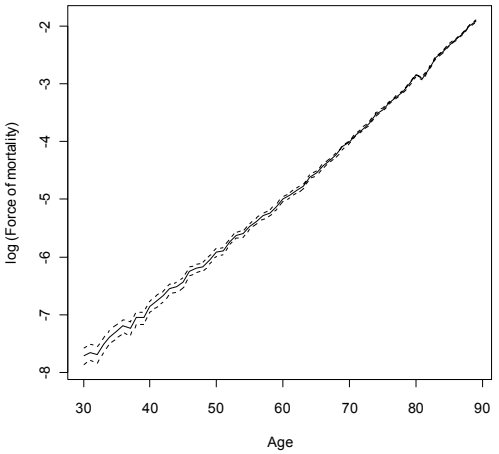
(ii)



(iii)



(iv)



2: Fitting a Gompertz-Makeham model for each calendar year to CMI male assured lives data and to England and Wales population data

2.1 Model

As stated in Chapter 1, the GM(r,s) model is defined by:

$$\mu_{xt} = \sum_{i=0}^{r-1} \kappa_t^{(i)} x^i + \exp\left(\sum_{j=0}^{s-1} \kappa_t^{(r+j)} x^j \right),$$

where μ_{xt} is the force of mortality at age x in calendar year t and $\kappa_t^{(0)}, \ldots, \kappa_t^{(r+s-1)}$ are parameters to be estimated for each t. x and t are considered to be discrete variables, *i.e.* we assume that mortality rates only change when lives attain a new age label or when a new calendar year begins.

We follow Brouhns *et al.* (2002) in assuming that the number of deaths at each age x in each calendar year t has a Poisson distribution with parameter E_{xt} multiplied by this μ_{xt}. The range of ages we consider is denoted $x_1,\ldots,x_N$. The largest values of r and s we shall consider are both 4. We also reparameterise the models so that as x varies for fixed t, the mean of the quantities to which each of the kappa parameters (other than $\kappa_t^{(0)}$ and $\kappa_t^{(r)}$) is applied is zero. This is done to avoid excessively large numbers appearing in the estimation process for the kappa parameters. Under the GM(3,4) model, for example, D_{xt} then has a Poisson distribution with parameter $\mu_{xt}E_{xt}$, where:

$$\begin{aligned} \mu_{xt} &= \kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \kappa_t^{(2)}((x-\bar{x})^2 - \hat{\sigma}_x^2) \\ &+ \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \kappa_t^{(6)}(x-\bar{x})^3]' \end{aligned}$$

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i ,$$

$$\hat{\sigma}_x^2 = \frac{1}{N}\sum_{i=1}^{N} (x_i - \bar{x})^2 .$$

In what follows, we shall always label the kappa parameters as in the above formula, even when we are considering values of r less than 3 and/or values of s less than 4. In the case where r = 4, we can no longer label the kappa parameters in this way, but we shall have no need to refer explicitly to the kappa parameters in any model with r = 4.

For comparison, the CBD model referred to in Chapter 1 is equivalent to a GM(0,2) model except that, instead of a linear function of age for each calendar year being fitted to log μ_{xt}, it is fitted to:

$$\log\left(\frac{q_{xt}}{1-q_{xt}} \right),$$

where q_{xt} is the initial mortality rate at age x in calendar year t.

2.2 Parameter estimation methodology

The Poisson assumption implies that the log-likelihood function of the full GM(3,4) model is:

$$\ell = c - \sum_x \sum_t E_{xt} \mu_{xt}(\kappa_t^{(0)}, \kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}, \kappa_t^{(4)}, \kappa_t^{(5)}, \kappa_t^{(6)})$$
$$+ \sum_x \sum_t D_{xt} \log \mu_{xt}(\kappa_t^{(0)}, \kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}, \kappa_t^{(4)}, \kappa_t^{(5)}, \kappa_t^{(6)}) \quad ,$$

where:

$$\mu_{xt}(\kappa_t^{(0)}, \kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}, \kappa_t^{(4)}, \kappa_t^{(5)}, \kappa_t^{(6)}) = \kappa_t^{(0)} + \kappa_t^{(1)}(x - \bar{x}) + \kappa_t^{(2)}((x - \bar{x})^2 - \hat{\sigma}_x^2)$$
$$+ \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x}) + \kappa_t^{(5)}((x - \bar{x})^2 - \hat{\sigma}_x^2) + \kappa_t^{(6)}(x - \bar{x})^3]$$

and c is a constant.

The estimation of the kappa parameters was by maximum likelihood. An iterative scheme was used, where each step consisted of updating the values of one of the kappa parameters for all calendar years t using the Newton-Raphson method, leaving the other kappa parameters constant. Brouhns *et al.* (2002) use a similar scheme to fit the model they consider to Belgian population mortality data. Further details of the iterative scheme are given in Appendix A.

Table 2.1 shows the parameter values that were taken as the starting point for the estimation of each of the GM models fitted. The iterations were stopped when a complete loop of the iterations, from one step of updating the $\kappa^{(0)}$ parameters to another, changed none of the parameter estimates by more than 10^{-6}. In other words, the iterations were stopped when, in the notation of Appendix A:

$$\left| \varepsilon_t^{(i)} - \alpha_t^{(i)} \right| < 10^{-6}$$

for all t and all i. An exception was that if this condition was satisfied at the very first loop of the iterations, then they were not stopped.

Table 2.1 – Starting points for the iterations to estimate the parameters of the GM models

Model	Starting point for iterations
GM(0,2)	Mortality rate independent of age for each calendar year – $\kappa_t^{(3)} = \log\left(\frac{\sum_x D_{xt}}{\sum_x E_{xt}}\right)$, $\kappa_t^{(4)} = 0$
GM(1,2)	Result of GM(0,2) model
GM(1,3)	Result of GM(1,2) model
GM(2,3)	Result of GM(1,3) model
GM(3,3) (fitted to CMI data only)	Result of GM(2,3) model
GM(2,4)	Result of GM(2,3) model
GM(3,4) (fitted to England and Wales data only)	Result of GM(2,4) model
GM(4,4) (fitted to England and Wales data only)	Result of GM(3,4) model

In all the models fitted, the $\kappa^{(3)}$ parameters were the ones of highest magnitude, so in practice, the iterations were stopped when the greatest absolute movement of any $\kappa^{(3)}$ parameter over a loop was just less than 10^{-6}.

It should be noted that there is no interaction between calendar years in this fitting procedure, other than in the criterion for stopping the iterations. We are therefore estimating the kappa parameters for each calendar year independently. We should like to be able to explain the kappa parameter graphs we obtain by reference to period effects alone. However, the next section shows that period effects alone do not always provide adequate explanations.

2.3 Parameter estimation results

2.3.1 CMI data

Graphs of the parameter estimates for each of the models fitted to CMI data are shown in Figures 2.1-2.6. These figures relate to the GM(0,2), GM(1,2), GM(1,3), GM(2,3), GM(3,3) and GM(2,4) models respectively.

The downward slope of the $\kappa^{(3)}$ parameters in Figure 2.1(i) indicates an overall improvement of mortality rates over the period 1947-2005, as noted in the 'Data' section of Chapter 1. The upward slope of the $\kappa^{(4)}$ parameters in Figure 2.1(ii) indicates that, in percentage terms, this improvement has been more rapid at younger ages than at older ages.

In the GM(1,2) model compared with the GM(0,2) model, a quantity, $\kappa_t^{(0)}$, independent of age x for each calendar year t has been added to the formula for μ_{xt}. Intuitively, one might expect these quantities to be positive, representing mortality from unnatural causes such as accidents and violence. However, Figure 2.2(i) shows that many of the fitted values of $\kappa_t^{(0)}$ are in fact negative. In fact, the years t for which this happens are from 1948 to 1988 inclusive, excluding 1985. This potentially gives rise to an issue if the model is extrapolated to ages under 30 (the youngest age in the data set), in that the extrapolated forces of mortality might be negative. This possibility was investigated and it was found that the extrapolated forces of mortality at small positive ages were indeed negative in the years 1950-84 inclusive and 1987. In certain years, the negative forces of mortality extended up to and including age 21. This indicates that, for the CMI data set, the GM(1,2) model should not be extrapolated to ages under 30 without adjustment.

Figures 2.2(ii) and (iii), compared with Figures 2.1(i) and (ii), respectively show that the overall trend of the $\kappa^{(3)}$ and $\kappa^{(4)}$ parameters with time is similar under the GM(1,2) model to that under the GM(0,2) model. Figure 2.2(ii) is almost identical to Figure 2.1(i). The differences between Figures 2.2(iii) and 2.1(ii) are slightly greater but the upward slope of the $\kappa^{(4)}$ parameters remains the key feature.

We see from Figure 2.3(i) that in the GM(1,3) model, unlike the GM(1,2) model, all the fitted $\kappa^{(0)}$ parameters are positive. This is consistent with the intuitive interpretation of the $\kappa^{(0)}$ parameters as representing mortality from unnatural causes. It means that, for this data set, the GM(1,3) model is more likely than the GM(1,2) model to be suitable for extrapolation to ages under 30, as all the extrapolated forces of mortality will be positive.

Figures 2.3(ii) and (iii) show, respectively, the fitted $\kappa^{(3)}$ and $\kappa^{(4)}$ parameters under the GM(1,3) model. As in the case of the GM(1,2) model compared with the GM(0,2) model, the overall trend has not changed in the GM(1,3) model compared with the GM(1,2) model for either the $\kappa^{(3)}$ or $\kappa^{(4)}$ parameters. Figure 2.3(ii) is almost identical to Figure 2.2(ii). The differences between Figures 2.3(iii) and 2.2(iii) are slightly greater but the upward slope of the $\kappa^{(4)}$ parameters remains the key feature.

The negative fitted $\kappa^{(5)}$ parameters under the GM(1,3) model shown in Figure 2.3(iv) indicate underlying mortality that increases more slowly than exponentially with age at the oldest ages.

According to the two statistical tests which we shall describe in Section 2.5, the GM(2,3) model achieves a statistically significant improvement in fit over the GM(1,3) model. However, inspection of the parameter graphs for the GM(2,3) model in Figure 2.4 reveals certain features which seem more likely to constitute evidence of overfitting rather than reflecting genuine features of the underlying mortality rates. Of particular note are the strong positive correlation between the fitted $\kappa^{(0)}$ and $\kappa^{(1)}$ parameters and the strong positive correlation between the $\kappa^{(3)}$ parameters, the negatives of the $\kappa^{(4)}$ parameters and the $\kappa^{(5)}$ parameters, together with the greatly increased range of values taken by each of the parameter series individually. The volatility of the $\kappa^{(3)}$ parameters in particular is also much increased compared with the GM(1,3) model. If such features are included in future projections, then the results are likely to be inappropriate. Note that the correlations might be eliminated by considering

linear combinations of the different kappa series that are uncorrelated, but this would give rise to the issue that there is no unique method of constructing such linear combinations.

Figures 2.4(i) and (ii), showing negative $\kappa^{(0)}$ and $\kappa^{(1)}$ parameters respectively, suggest that the GM(2,3) model, like the GM(1,2) model, has the potential to give negative forces of mortality when extrapolated to ages under 30. This was investigated and was found to be an issue for the calendar years 1998 and 2000-05 inclusive. The year for which this issue is most significant is 2004, with the extrapolated forces of mortality being negative up to and including age 25. This indicates that for this data set, the GM(2,3) model, like the GM(1,2) model, is not suitable for extrapolation to ages under 30 without adjustment.

Figure 2.5 shows that the fitted parameter values of the GM(3,3) model, as for the GM(2,3) model, have certain features which suggest that there has been overfitting. There are strong positive correlations between the $\kappa^{(0)}$, $\kappa^{(1)}$ and $\kappa^{(2)}$ parameters and between the $\kappa^{(3)}$ parameters, the negatives of the $\kappa^{(4)}$ parameters and the $\kappa^{(5)}$ parameters, and the ranges of values taken by the parameters are even wider than under the GM(2,3) model. The shape of the graph of $\kappa^{(3)}$ parameters is difficult to interpret as one would expect a steady improvement in mortality with time, and hence a steady fall in the $\kappa^{(3)}$ parameters.

Figure 2.6, showing graphs of the parameter estimates for the GM(2,4) model, has parameter values for 1955 and 1956 that are very different from those for adjacent years. It is not apparent what feature of the data is causing these differences, but in general, when a model is overfitted, issues can arise in that small changes in the input data can lead to significant changes in the parameter estimates. Figure 2.6 also shows strong positive correlation between the $\kappa^{(3)}$ parameters, the negatives of the $\kappa^{(4)}$ parameters, the $\kappa^{(5)}$ parameters and the negatives of the $\kappa^{(6)}$ parameters, again providing evidence of overfitting.

2.3.2 England and Wales male data

Figures 2.7, 2.8, 2.9 and 2.10 respectively show the parameter estimates for the GM(0,2) model, the GM(1,2) model, the GM(1,3) model and the GM(2,3) model fitted to England and Wales male data. For models more complex than the GM(2,3) model, the corresponding graphs have not been shown because overfitting is already apparent in the graphs for the GM(2,3) model.

As for Figure 2.1, Figure 2.7 shows that there is an overall trend of improving mortality with time and that the improvements have tended to be faster in percentage terms at younger ages than at older ages. Comparing Figure 2.7 with Figure 2.1, there is evidence that the rate of improvement has been lower for England and Wales male data than for CMI data, although the difference in scale on the time axis are should be borne in mind. As the CMI data is weighted towards the higher socio-economic groups, this is consistent with the observation that mortality has improved more quickly among higher than among lower socio-economic groups. See Willets *et al*. (2004). The values of the $\kappa^{(4)}$ parameters are lower for the England and Wales male data than for the CMI data, indicating that these mortality differentials between socio-economic groups narrow with increasing age.

In Figure 2.8, we again have a downward-sloping graph of $\kappa^{(3)}$ parameters and an upward-sloping graph of $\kappa^{(4)}$ parameters. It is not clear how the graph of $\kappa^{(0)}$ parameters should be interpreted – it is likely that it reflects effects that are more appropriately incorporated in the model by introducing $\kappa^{(5)}$ parameters. As in Figure 2.2, the negative values of the earlier $\kappa^{(0)}$ parameters will lead to negative forces of mortality if the model is extrapolated below age 30, but in more recent years the $\kappa^{(0)}$ parameters have been positive.

Figure 2.9 shows that, as for CMI data, introducing $\kappa^{(5)}$ parameters to the model for England and Wales male data makes all the $\kappa^{(0)}$ parameters positive. However, the most striking features of Figure 2.9 are that the $\kappa^{(0)}$ and $\kappa^{(4)}$ parameters rise sharply from 1980 to the early 1990s and then fall again, the $\kappa^{(3)}$ parameters fall more steeply than usual in the 1980s and early 1990s and the $\kappa^{(5)}$ parameters rise sharply from the mid-1990s onwards. Similarly to the graphs of the fitted parameter values in Cairns *et al*. (2006), the shapes of the graphs in Figure 2.9 can be explained by the cohort effect described in Chapter 1. Lives in England and

Wales born between 1925 and 1945, and particularly around 1931, have experienced significantly lower mortality than the preceding generation, to an extent not explained by the overall improvement of mortality with time. In the 1980s and early 1990s, these lives were mostly in their 50s, and so a significant improvement in their mortality but not in the mortality of older generations might be expected to lead to a sharper than usual fall in the general level of the mortality curve, as measured by the $\kappa^{(3)}$ parameters, and to an increase in the slope of the mortality curve, as measured by the $\kappa^{(4)}$ parameters. As the $\kappa^{(4)}$ parameters have increased, the $\kappa^{(0)}$ parameters will then tend to increase to maintain broadly the same level of mortality at the youngest ages. This is indeed what we observe. Between the early 1990s and 2005, these lives were mostly in their 60s and early 70s, and so we might expect a decrease in the slope of the mortality curve, as measured by the $\kappa^{(4)}$ parameters, and an increase in mortality at the oldest ages relative to mortality in the 60s and early 70s of age, as measured by the $\kappa^{(5)}$ parameters. Again we do in fact observe this. We also observe the increases in the $\kappa^{(0)}$ parameters that occurred in the 1980s reversing out between the early 1990s and 2005.

As for Figure 2.4, Figure 2.10 suggests that the introduction of $\kappa^{(1)}$ parameters constitutes overfitting. In the GM(2,3) model fitted to England and Wales male data, there is strong positive correlation between the $\kappa^{(0)}$ parameters, the $\kappa^{(1)}$ parameters, the negatives of the $\kappa^{(3)}$ parameters, the $\kappa^{(4)}$ parameters and the negatives of the $\kappa^{(5)}$ parameters, and the range of variation of the $\kappa^{(4)}$ parameters in particular has increased significantly compared with the GM(1,3) model. As for the GM(2,3) model fitted to CMI data, the GM(2,3) model fitted to England and Wales male data was also extrapolated below age 30. Some issues with the extrapolated forces of mortality being negative were found, although fewer than for CMI data.

2.3.3 England and Wales female data

Figures 2.11, 2.12, 2.13 and 2.14 respectively show the parameter estimates for the GM(0,2) model, the GM(1,2) model, the GM(1,3) model and the GM(2,3) model fitted to England and Wales female data. For models more complex than the GM(2,3) model, the corresponding graphs have again not been shown.

Figure 2.11(i), showing the $\kappa^{(3)}$ parameters under the GM(0,2) model for female data, is slightly less steep than Figure 2.7(i), for male data, indicating that female mortality has improved slightly less rapidly than male mortality over the period 1962-2005, although the majority of the mortality differential between the genders that was present in 1962 remains in 2005. The higher values of the $\kappa^{(4)}$ parameters for the GM(0,2) model fitted to England and Wales female data, as shown in Figure 2.11(ii), than fitted to England and Wales male data, as shown in Figure 2.7(ii), indicate that the mortality differential between the genders narrows with increasing age, although this effect is not significant enough to close the gap between the genders completely at the oldest ages in the data.

The key differences between Figure 2.12 and Figure 2.8 are that, between 1962 and the late 1970s, the $\kappa^{(0)}$ parameters for females decreased whereas those for males increased, the $\kappa^{(3)}$ parameters for females decreased rather more slowly than for males, and the $\kappa^{(4)}$ parameters for females decreased whereas those for males increased. Similar (indeed, stronger) features can also be seen in Figure 2.13 and will be discussed below.

The discussion of the impact of cohort effects on Figure 2.9 is also relevant to Figure 2.13. However, in the case of Figure 2.13, the shapes of the graphs between 1962 and the late 1970s also require explanation. It seems likely that the graphs reflect a further cohort effect, but in the direction of low rather than high mortality improvements for a particular generation. Table 1b and Figure 1b of Willets (2004) indicate that females born around 1915 have experienced particularly low rates of mortality improvement over the previous generation. This is likely to be mainly a result of an increase in smoking prevalence – Figure 5 of the same paper shows that lifetime cigarette consumption for females as a function of year of birth was increasing most rapidly around 1915.

Two further features of Figure 2.13 which did not occur in the corresponding figures for either of the male data sets should be noted. Firstly, most of the fitted $\kappa^{(0)}$ parameters are negative and this may lead to negative forces of mortality if the GM(1,3) model fitted to England and

Wales female data is extrapolated to ages below 30. This extrapolation was performed and it was found that in 26 of the 44 years from 1962 to 2005, the extrapolated force of mortality at age 0 was negative. In certain years the negative extrapolated forces of mortality extended up to and including age 21. This shows that the GM(1,3) model fitted to this data set should not be extrapolated below age 30 without adjustment.

The second feature of Figure 2.13 that should be noted is that, unlike either of the male data sets, the fitted $\kappa^{(5)}$ parameters are all positive, indicating mortality that increases more rapidly than exponentially with age at the oldest ages. This increase is offset to some extent by the lower values of the fitted $\kappa^{(4)}$ parameters than for the male data sets.

On the basis of Figure 2.14, the GM(2,3) model appears to be overfitted to England and Wales female data. There is strong positive correlation between the $\kappa^{(0)}$ parameters, the $\kappa^{(1)}$ parameters, the negatives of the $\kappa^{(3)}$ parameters, the $\kappa^{(4)}$ parameters and the negatives of the $\kappa^{(5)}$ parameters, and the ranges over which the parameter values vary (except for the $\kappa^{(0)}$ parameters) have increased substantially compared with the GM(1,3) model. This is the same conclusion that we reached when fitting the GM(2,3) model to the male data sets.

2.4 Results for the force of mortality

Under the GM(0,2) model, the graph of the logarithm of the fitted force of mortality against age for a given calendar year is a straight line. Figures 2.15-2.17 show the differences between the fitted forces of mortality under the more complex models and these straight lines for a typical calendar year, 2000.

The graphs labelled (ii) in Figures 2.15-2.17 show that, for all three data sets, by far the largest impact of moving from the GM(0,2) model to the GM(1,2) model is to increase the fitted force of mortality at younger ages. This is consistent with our observation in the discussion of Figures 1.4-1.6 that the percentage rate of increase of mortality with age slows down at the younger ages in the data. The graphs labelled (ii) also show that moving from GM(0,2) to GM(1,2) decreases fitted mortality over the approximate age range 50-75 and increases it at ages over approximately 75. It is not clear at this stage whether these changes improve the fit or whether they arise because of the restricted range of shapes of the mortality curve available under the GM(1,2) model.

The key changes in the graphs labelled (iii) in Figures 2.15-2.17, for the GM(1,3) model, compared with the graphs labelled (ii) are towards the top of the age range of the data, which is not surprising given that the $\kappa^{(5)}$ parameter has more impact on fitted mortality towards the top of the age range than at other ages. For the male data sets, the negative $\kappa^{(5)}$ parameter tends to reduce mortality at the oldest ages, but for England and Wales female data, the positive $\kappa^{(5)}$ parameter tends to increase mortality at the oldest ages. The graphs give us no reason to doubt that the decrease in fitted mortality over the age range 50-75 under the GM(1,2) model compared with the GM(0,2) model did genuinely improve the fit, but they do suggest that for both the male data sets, the increase at ages over 75 was caused mainly by the restricted range of shapes under the GM(1,2) model. We shall consider the goodness of fit of the models at different ages further in Section 2.8.

The graphs in Figures 2.15-2.17 for models more complex than the GM(1,3) model show no significant qualitative changes in shape compared with the graphs for the GM(1,3) model, with the exception of the very youngest ages in England and Wales female data where there is a decrease in mortality compared with the GM(1,3) model. Thus Figures 2.15-2.17 do not provide any strong reason to support the use of a more complex model than GM(1,3).

2.5 Akaike Information Criterion (AIC) and Bayes Information Criterion (BIC)

We should like a model with enough parameters to give a good fit to the data. However, we should not like to complicate the model unnecessarily by introducing parameters which the data do not provide sufficient evidence to conclude are necessary. Two quantitative criteria that have been developed to inform the judgement as to this balance between goodness of fit

and simplicity are the *Akaike Information Criterion (AIC)* and the *Bayes Information Criterion (BIC)*, also called the *Schwarz-Bayes Criterion (SBC)*. See, for example, Cairns (2000).

The AIC of a model is defined as:

$$AIC = \ell - k\,,$$

where ℓ is the maximum log-likelihood of the model and k is the number of parameters in the model. Models with a high AIC are to be preferred to those with a low AIC. The AIC is derived from the classical problem of testing the null hypothesis that the data are adequately described by a simple model M_1 against the alternative of a more complex model, M_2, which contains M_1 as a special case. The AIC does not require us to formulate any subjective prior beliefs as to which of the proposed models are most likely to be appropriate, and this might be thought to be an advantage. However, a disadvantage of the AIC is that its derivation assumes that the models being compared are nested, and so does not address the question of whether it is appropriate to use the AIC to compare two models that are not nested.

The BIC of a model is defined as:

$$BIC = \ell - \tfrac{1}{2} k \log N\,,$$

where ℓ and k are as above and N is the number of data cells, *i.e.* N = 3,599 for CMI data and 2,640 for England and Wales data. Models with a high BIC are to be preferred to those with a low BIC. The BIC is derived using Bayesian theory, starting from a prior belief that all the models under consideration are equally likely to be the correct one and comparing the posterior probabilities of the different models. Note that the BIC penalises complexity more severely than the AIC, particularly when, as in this case, N is large. The prior assumption that all models are equally likely could be modified subjectively if this were considered necessary, by assigning different prior probabilities to different models, but such subjectivity might still be considered undesirable.

Comparisons of the AIC and BIC between different models can be used to gain an indication of how significant certain features of the data are as well as to inform the choice of model. Therefore, in the tables below, we shall also show the AIC and BIC of the 'static GM(0,2)' model, in which the $\kappa^{(3)}$ and $\kappa^{(4)}$ parameters do not depend on calendar year, so that the force of mortality at age x for all calendar years t is:

$$\mu_{xt} = \exp[a + b(x - \bar{x})]\,,$$

where a and b are parameters to be determined. The fitted values of a and b are shown in Table 2.2.

Table 2.2 – Fitted values of the parameters a and b for the static GM(0,2) model

Data set	a	b
CMI	-4.5348	0.10410
E&W Male	-4.1818	0.09571
E&W Female	-4.7492	0.10075

The improvement in the AIC and BIC from the static GM(0,2) model to the GM(0,2) model, compared with the improvements from the GM(0,2) model to the more complex GM models, will give an indication of the relative significance of the improvement of mortality over time and the nonlinearity of the shape of the graph of log mortality against age.

Table 2.3 shows the values of both the AIC and BIC for the GM models that were fitted to the CMI data.

Table 2.3 – AIC and BIC values for a number of possible GM models fitted to CMI data

Model	Maximum log-likelihood	Number of parameters	AIC	BIC
Static GM(0,2)	-65,320.81	2	-65,322.81	-65,329.00
GM(0,2)	-19,237.78	118	-19,355.78	-19,720.90
GM(1,2)	-18,536.30	177	-18,713.30	-19,260.98
GM(1,3)	-16,644.01	236	-16,880.01	-17,610.25
GM(2,3)	-16,257.82	295	-16,552.82	-17,465.61
GM(3,3)	-16,202.21	354	-16,556.21	-17,651.56
GM(2,4)	-16,181.63	354	-16,535.63	-17,630.98

We conclude from Table 2.3 that each of the models up to and including GM(2,3) achieves a statistically significant improvement in fit over the previous one, under both the BIC and the AIC. However neither GM(3,3) nor GM(2,4) achieves a statistically significant improvement over GM(2,3) under the BIC, and only GM(2,4) achieves a marginal improvement over GM(2,3) under the AIC.

We also conclude from Table 2.3 that the improvement of mortality over time is a much more significant feature of the data than nonlinearity of the graph of log mortality against age, as moving from the static GM(0,2) model to the GM(0,2) model increases the AIC and BIC much more than moving from the GM(0,2) model to the more complex models. This suggests that there is unlikely to be justification in any application for failing to allow for the improvement of mortality over time. Failure to allow for this feature will result in unduly low reserves and prices being calculated for annuity business and, potentially, unduly high reserves and prices for assurance business. However, in applications where only an approximate result is required, it may be appropriate to assume that mortality is a linear function of age for each calendar year with parameters depending on the year, *i.e.* to use the GM(0,2) model.

The fact that the GM(2,3) model achieves a statistically significant improvement in fit over the GM(1,3) model, under both the BIC and the AIC, demonstrates that there is more structure in the data than is reflected in the GM(1,3) model. However, this does not necessarily imply that the GM(2,3) model is the most appropriate model to use for future projections. In general, if the fitted parameters under a model show features which we are unable to interpret, then we are unable to assess whether or not these features are likely to be repeated in the future, and hence whether or not it is appropriate to include them in future projections. In the case of the GM(2,3) model, interpretation of the changes in the kappa parameter values over time is significantly complicated by the strong correlations between them, and this would tend to support the use of the GM(1,3) model rather than the GM(2,3) model for projections.

Table 2.4 shows the values of the AIC and BIC for the GM models that were fitted to England and Wales male data, and Table 2.5 shows the values for England and Wales female data.

Table 2.4 – AIC and BIC values for a number of possible GM models fitted to England and Wales male data

Model	Maximum log-likelihood	Number of parameters	AIC	BIC
Static GM(0,2)	-309,023.91	2	-309,025.91	-309,031.79
GM(0,2)	-47,095.64	88	-47,183.64	-47,442.29
GM(1,2)	-34,306.16	132	-34,438.16	-34,826.14
GM(1,3)	-18,478.49	176	-18,654.49	-19,171.81
GM(2,3)	-17,751.70	220	-17,971.70	-18,618.34
GM(2,4)	-17,181.03	264	-17,445.03	-18,221.00
GM(3,4)	-16,937.19	308	-17,245.19	-18,150.48
GM(4,4)	-16,785.46	352	-17,137.46	-18,172.08

Table 2.5 – AIC and BIC values for a number of possible GM models fitted to England and Wales female data

Model	Maximum log-likelihood	Number of parameters	AIC	BIC
Static GM(0,2)	-177,258.66	2	-177,260.66	-177,266.54
GM(0,2)	-26,688.23	88	-26,776.23	-27,034.89
GM(1,2)	-21,918.09	132	-22,050.09	-22,438.07
GM(1,3)	-18,911.86	176	-19,087.86	-19,605.17
GM(2,3)	-17,705.78	220	-17,925.78	-18,572.42
GM(2,4)	-16,411.35	264	-16,675.35	-17,451.32
GM(3,4)	-16,140.94	308	-16,448.94	-17,354.23
GM(4,4)	-15,872.44	352	-16,224.44	-17,259.06

The GM(4,4) model was found to be the most complex model for which the iterative scheme converged, for both male and female England and Wales data.

Given the greater size of the England and Wales data sets than of the CMI data set, it was expected that more parameters would be statistically significant in Tables 2.4 and 2.5 than in Table 2.3. In fact, for both the male and the female England and Wales data sets, the most complex model we have fitted, the GM(4,4) model, has been determined to be optimal under both the AIC and the BIC, with the exception that the BIC of the GM(4,4) model fitted to male data is slightly less than that of the GM(3,4) model. However, the improvement of mortality over time is again a much more significant feature of the data than nonlinearity of the graph of log mortality against age.

As with the CMI data, the conclusion to be drawn from the fact that more complex models can improve the AIC and BIC compared with the GM(1,3) model is that there is additional structure in the data, rather than that it is appropriate to base projections on those more complex models. For similar reasons to those for the CMI data, the GM(1,3) model is more likely than the GM(2,3) model to be an appropriate basis for projections. However, as consideration of cohort effects was necessary to explain the shapes of the parameter graphs under the GM(1,3) model for both males and females, we should also investigate models that make explicit allowance for cohort effects. This will be the subject of Chapter 3.

2.6 Standardised residuals

For a given model fitted to a given data set, the standardised residual at age *x* in calendar year *t*, Z_{xt}, is defined as:

$$\frac{D_{xt} - \hat{\mu}_{xt} E_{xt}}{\sqrt{\hat{\mu}_{xt} E_{xt}}},$$

where $\hat{\mu}_{xt}$ is the fitted force of mortality at age *x* in calendar year *t*. The $\hat{\mu}_{xt} E_{xt}$ in the numerator represents the mean of the fitted Poisson distribution of the number of deaths at age *x* in calendar year *t* and the $\hat{\mu}_{xt} E_{xt}$ in the denominator represents the variance of this Poisson distribution.

If the model under consideration were a perfect representation of reality, then the standardised residuals would be realisations of independent random variables with mean 0 and variance 1. In practice, the sample variance of the standardised residuals would be expected to be greater than 1, as there will always be some patterns of variation in the data that are not captured by the model. Table 2.6 shows the sample variances of the standardised residuals observed for each of the GM models fitted to the CMI data. These have been calculated as:

$$\frac{1}{N-k}\sum_x\sum_t (Z_{xt}-\bar{z})^2,$$

where N is the number of data cells, k is the number of parameters and $\bar{z}$ is the sample mean of the standardised residuals, rather than as:

$$\frac{1}{N}\sum_x\sum_t (Z_{xt}-\bar{z})^2,$$

i.e. they are unbiased estimates rather than maximum likelihood estimates of the population variance of the standardised residuals.

Table 2.6 – Sample variances of the standardised residuals for a number of possible GM models fitted to CMI data

Model	Sample variance of standardised residuals
Static GM(0,2)	28.3338
GM(0,2)	3.5309
GM(1,2)	3.1662
GM(1,3)	2.0829
GM(2,3)	1.8843
GM(3,3)	1.8847
GM(2,4)	1.8723

It can be seen from Table 2.6 that the fit, as measured by the sample variance of the standardised residuals, improves significantly as we move from the GM(0,2) model to the GM(1,2) model and, particularly, from the GM(1,2) model to the GM(1,3) model. The improvement from the GM(1,3) model to the GM(2,3) model is rather smaller. There is very little improvement beyond the GM(2,3) model – in fact there is a deterioration for the GM(3,3) model compared with the GM(2,3) model as the decrease in the sum of squares is not sufficient to outweigh the increase in the number of parameters.

Table 2.7 shows the sample variances of the standardised residuals observed for each of the GM models fitted to the England and Wales male data, and Table 2.8 shows the same information for the England and Wales female data.

Table 2.7 – Sample variances of the standardised residuals for a number of possible GM models fitted to England and Wales male data

Model	Sample variance of standardised residuals
Static GM(0,2)	221.3328
GM(0,2)	26.4706
GM(1,2)	17.2309
GM(1,3)	4.5875
GM(2,3)	4.0648
GM(2,4)	3.6561
GM(3,4)	3.5154
GM(4,4)	3.4497

Table 2.8 – Sample variances of the standardised residuals for a number of possible GM models fitted to England and Wales female data

Model	Sample variance of standardised residuals
Static GM(0,2)	124.5906
GM(0,2)	11.3289
GM(1,2)	7.5568
GM(1,3)	5.2179
GM(2,3)	4.3146
GM(2,4)	3.2973
GM(3,4)	3.1266
GM(4,4)	2.9505

The figures in Tables 2.7 and 2.8 are higher than the corresponding figures for the same model in Table 2.6. This is to be expected because the greater size of the England and Wales data sets than of the CMI data set makes it likely that more features of the data not captured by the model are will be statistically significant in the former than in the latter. It is also likely that the higher figures in Tables 2.7 and 2.8 than in Table 2.6 are partly due to lower accuracy of the exposed to risk in the England and Wales data than in the CMI data. The CMI calculates the exposed to risk in its data sets every year, whereas a census of the England and Wales population is only carried out every ten years, with the exposed to risk being estimated in intermediate years.

Tables 2.7 and 2.8 show very significant improvements in fit for both males and females as we move from the GM(0,2) to the GM(1,2) model, and as we move from the GM(1,2) to the GM(1,3) model. From the GM(1,3) to the GM(2,3) model and from the GM(2,3) to the GM(2,4) model, the improvement in fit is fairly significant for females, but less so for males. Beyond the GM(2,4) model, the improvements are relatively small for both males and females.

2.7 Coefficient of determination, R^2

For each of the data sets, it is instructive to examine what proportion of the variation in the crude force of mortality between data cells is explained by each of the models. The measure generally used for this purpose is called the *coefficient of determination* and is denoted by R^2.

We define the *residual sum of squares* (*RSS*) to be the sum of the squares of the standardised residuals:

$$RSS = \sum_x \sum_t Z_{xt}^2 \ .$$

We also define:

$$RSS_0 = \sum_x \sum_t \frac{\left(D_{xt} - \dfrac{D_{++} E_{xt}}{E_{++}} \right)^2}{\dfrac{D_{++} E_{xt}}{E_{++}}},$$

where:

$$D_{++} = \sum_x \sum_t D_{xt}$$

and:

$$E_{++} = \sum_x \sum_t E_{xt} .$$

Thus RSS_0 is the RSS of the model in which μ_{xt} is a constant, μ, independent of both age x and calendar year t. The maximum likelihood estimate of μ under this model is $\frac{D_{++}}{E_{++}}$.

R^2 is then defined as:

$$R^2 = 1 - \frac{RSS}{RSS_0} .$$

Thus $R^2 = 1$ in a model where the graduated force of mortality equals the observed force of mortality exactly in every data cell, and $R^2 = 0$ if $\hat{\mu}_{xt} = \frac{D_{++}}{E_{++}}$ for all x and all t.

The values of R^2 for the different models fitted to CMI data, to England and Wales male data and to England and Wales female data are shown in Tables 2.9, 2.10 and 2.11 respectively. In Tables 2.9, 2.10 and 2.11, 'Flat' represents the model where μ_{xt} is independent of both age x and calendar year t.

Table 2.9 – Values of R^2 for a number of possible GM models fitted to CMI data

Model	R^2
Flat	0
Static GM(0,2)	0.981448
GM(0,2)	0.997763
GM(1,2)	0.998028
GM(1,3)	0.998725
GM(2,3)	0.998867
GM(3,3)	0.998887
GM(2,4)	0.998894

Table 2.10 – Values of R^2 for a number of possible GM models fitted to England and Wales male data

Model	R^2
Flat	0
Static GM(0,2)	0.981410
GM(0,2)	0.997849
GM(1,2)	0.998624
GM(1,3)	0.999640
GM(2,3)	0.999687
GM(2,4)	0.999723
GM(3,4)	0.999739
GM(4,4)	0.999749

Table 2.11 – Values of R^2 for a number of possible GM models fitted to England and Wales female data

Model	R^2
Flat	0
Static GM(0,2)	0.989749
GM(0,2)	0.999098
GM(1,2)	0.999409
GM(1,3)	0.999599
GM(2,3)	0.999674
GM(2,4)	0.999756
GM(3,4)	0.999773
GM(4,4)	0.999789

For all three data sets, by far the greatest increase in R^2 occurs when we move from the flat model to the static GM(0,2) model. In each case, over 98% of the variation in the crude force of mortality between data cells is explained when we move from assuming that mortality is independent of age to assuming that it increases exponentially with age, without making any allowance for the improvement of mortality over time.

The GM(0,2) model, in which mortality increases exponentially with age for each calendar year with different parameters for each calendar year, accounts for over 99.75% of the variation in the crude force of mortality between data cells for all three data sets. However, we saw in Section 2.5 that there is still sufficient structure in the residuals of the GM(0,2) model to justify a more complex model, as more complex models improved the BIC and AIC compared with the GM(0,2) model.

Although Section 2.3.1 concluded that models more complex than the GM(1,3) model were overfitted to CMI data, the GM(1,3) model has a rather lower value of R^2 than for England and Wales data. This can be attributed to the fact that, as the CMI data is a smaller data set than the England and Wales data sets, more of the variation in the crude force of mortality between data cells is a result of stochastic fluctuations.

2.8 Residual plots

For each of the GM models fitted to each of the data sets, it is instructive to examine which ages and calendar years have observed mortality rates above the modelled mortality rates (*i.e.* positive standardised residuals), and which have observed mortality rates below the modelled mortality rates (*i.e.* negative standardised residuals). For CMI data, these ages and years are shown respectively as red and blue in the graphs in Figure 2.18. Darker shades of red and blue are used to highlight the more extreme residuals.

The key observations from Figure 2.18(i) are that the GM(0,2) model systematically understates mortality at the very youngest ages of the data set, systematically overstates it around ages 35-50, systematically understates it around ages 55-65 and systematically overstates it above age 80. It is clear that modelling the force of mortality as an exponential of a linear function of age for each calendar year is inadequate to capture the structure of the data set. There is some evidence of diagonal patterns in the residuals, in that the age bands in which there is systematic understatement and systematic overstatement increase towards the end of the period 1947-2005. It seems likely that much of this reflects the favourable cohort effect centred on 1926. There is also a possible adverse cohort effect for lives born after 1945 – as discussed below, a similar effect is also visible for England and Wales male data.

Figure 2.18(ii) shows that the addition of $\kappa^{(0)}$ parameters is not in itself sufficient to eliminate the above inadequacies in the GM(0,2) model. The GM(1,2) model shows systematic discrepancies between observed and fitted mortality at broadly the same ages as the GM(0,2) model. This is despite the statistically significant improvement the GM(1,2) model showed in goodness of fit, under both the BIC and AIC, compared with the GM(0,2) model. There is still

some evidence, though slightly weaker than for the GM(0,2) model, of diagonal patterns in the residuals.

The generally paler colour of Figures 2.18(iii) and (iv) compared with Figures 2.18(i) and (ii) indicates that the systematic overstatements of mortality at certain ages and the systematic understatements at others are much reduced in the GM(1,3) model and, particularly, the GM(2,3) model, compared with the GM(1,2) model. However, one feature that remains is a systematic understatement of mortality around ages 60-65 and a systematic overstatement around ages 65-70. As we shall see later in this section, this feature is specific to the CMI data and is not observed with England and Wales data. A plausible explanation for this feature is that the policies around ages 60-65 are mostly endowments maturing at or near age 65, whereas the policies around ages 65-70 are mostly whole life assurances. The life insurance companies that contributed to the CMI's investigation may have sold endowments and whole life assurances to different market segments, possibly with stricter underwriting for whole life assurances, or the understatement of mortality in the CMI data resulting from unclaimed policies may have been more significant for whole life than for endowment business. As shown in Figure 1.1(i), the exposed to risk in the CMI data decreases rapidly with increasing age around age 65, which is consistent with this explanation. The author contacted the CMI regarding this feature and received the following response:

'As far as we are aware this feature has not previously been investigated or brought to the attention of the CMI. Unfortunately, the scheduled nature of the data provided to the CMI means that we are unable to give firm answers on why this might be occurring, for example we cannot separately identify endowment business and whole of life business. However, most endowment business would be written to mature by age 65 and so it may be reasonable to treat the data after age 65 as arising mainly from whole life policies.'

This feature of the CMI data has, however, become less significant since the early 1990s.

The diagonal patterns that we observed in Figures 2.18(i) and (ii) have been largely eliminated in Figures 2.18(iii) and (iv). This suggests that cohort effects in the CMI data are largely captured indirectly via the parameter estimates of the GM(1,3) and GM(2,3) models. When we discussed the parameter graphs in Section 2.3, we did not identify cohort effects as a significant driver of the shapes of the graphs. However, further inspection of Figure 2.3 in particular reveals a particularly steep fall in the $\kappa^{(3)}$ parameters, and a particularly steep rise in the $\kappa^{(4)}$ parameters, from 1980 to the early 1990s. In Section 2.3 we explained a similar feature of the England and Wales data sets by reference to cohort effects.

Figures 2.18(v) and (vi) reveal very little change compared with Figure 2.18(iv). This is consistent with the fact that the GM(3,3) model failed to improve the BIC or AIC compared with the GM(2,3) model, and the GM(2,4) model failed to improve the BIC and improved the AIC only marginally.

Figures 2.19(i), (ii), (iii) and (iv) are similar residual plots for the England and Wales male data, for the GM(0,2), GM(1,2), GM(1,3) and GM(2,3) models respectively. The residual plots for more complex models than GM(2,3) have not been shown, as we concluded in Section 2.3 that even the GM(2,3) model was overfitted. Figures 2.20(i)-(iv) are the corresponding plots for the female data.

The standardised residuals for a given model tend to be larger in magnitude for England and Wales data than for CMI data. This is again a consequence of the larger population underlying the England and Wales data than underlying the CMI data.

Cohort effects, represented by diagonal bands, are much more apparent in the residual plots for England and Wales data than for CMI data. As mentioned in Section 1 and in Section 2.3, the cohort centred on year of birth 1931 has been identified as exhibiting particularly large mortality improvements compared with the previous generation, for both England and Wales males and England and Wales females. In fact, if we draw the diagonals on Figures 2.19 and 2.20 representing year of birth 1931, then we find that the area close to and above the diagonals tends to be red, representing observed mortality above the modelled mortality for

years of birth just before 1931, and the area close to and below the diagonals tends to be blue, representing observed mortality below the modelled mortality for years of birth just after 1931. This is consistent with what we would expect if the cohort centred on year of birth 1931 exhibits particularly large mortality improvements compared with the previous generation.

Section 2.3 also mentioned that females born around 1915 have exhibited particularly low mortality improvements compared with the previous generation. In fact, if we draw the diagonals representing year of birth 1915 on Figure 2.20, then we find that the area close to and above the diagonals is mostly blue, while the area close to and below the diagonals is red or lighter blue. This is consistent with females born around 1915 exhibiting particularly low mortality improvements compared with the previous generation.

One feature that can be seen in all the residual plots for England and Wales data, for both males and females, is that lives born in 1919 have exhibited unusually low mortality and lives born in 1920 have exhibited unusually high mortality. Renshaw and Haberman (2006) observe a similar feature and attribute it to the 1919 influenza epidemic. It seems plausible that this epidemic could have had either a favourable impact on the mortality of the cohort *born* during or shortly after the epidemic, by reducing the chances of unhealthy children being born alive and/or surviving past infancy, or an adverse impact, because the epidemic had a permanent impact on the health of this cohort. However, it is by no means clear that the epidemic would have had a favourable impact on the mortality of lives born in 1919 and an adverse impact on the mortality of lives born in 1920. This is an area that would benefit from further research.

Figure 2.19(i) also suggests a possible adverse cohort effect for England and Wales males born since 1945. This feature is no longer apparent in Figure 2.19(ii), having been captured indirectly via $\kappa^{(0)}$ parameters that increase over time. Paragraph 2.15.6 of Willets *et al.* (2004) suggests the possible existence of such an adverse cohort effect for some generations born since 1945, but it cannot be said that there is sufficient evidence to demonstrate its existence definitively.

We shall now discuss each of the graphs in Figures 2.19 and 2.20 individually. From Figure 2.19(i), we see that the ages at which the GM(0,2) model systematically overstates and understates England and Wales male mortality are fairly similar to those for CMI data. However, with England and Wales male data there is no systematic understatement of mortality at the very youngest ages of the data set in the early years of the period 1962-2005, although such an understatement does appear in the later years. The systematic understatement at ages 55-65 that existed in CMI data has moved to the age range 60-75. There is stronger evidence than for CMI data of cohort effects giving rise to increases in the age bands of systematic overstatement and understatement in the later years of the data set.

Figure 2.19(ii) shows that the GM(1,2) model fitted to England and Wales male data systematically overstates and understates mortality at broadly the same ages as the GM(0,2) model. However, there is now a systematic understatement at the youngest ages in the earlier years as well as the later years of the data set, and elsewhere there is a limited amount of weakening of both the systematic overstatements and understatements.

Figure 2.19(iii) is significantly paler than Figures 2.19(i) and (ii), and the systematic overstatements and understatements of mortality at certain ages have largely been eliminated, indicating that the GM(1,3) model explains significantly more of the structure of the England and Wales male data than the GM(0,2) and GM(1,2) models. Some diagonal patterns remain, indicating cohort effects. We observed in Section 2.3 that the parameter estimates reflected the impact of cohort effects indirectly, but Figure 2.19(iii) shows that the parameter estimates captured only part of the cohort effects.

A comparison of Figure 2.19(iv) with Figure 2.19(iii) reveals some differences in specific regions, but it is unlikely that such a comparison would be used to justify the use of the GM(2,3) model in preference to the GM(1,3) model for England and Wales male data.

Comparing Figures 2.20(i) and (ii) with Figures 2.19(i) and (ii) shows that the ages for which the GM(0,2) and GM(1,2) models systematically overstate and understate England and Wales female mortality are broadly the opposite to those for England and Wales male mortality. In other words, mortality is systematically understated over the age range 35-50, systematically overstated over the age range 60-75 and systematically understated above age 80. As we move from the GM(0,2) to the GM(1,2) model, a systematic understatement of mortality at the youngest ages of the data set is replaced by a systematic overstatement.

The observations from Figures 2.20(iii) and (iv) are similar to those from Figures 2.19(iii) and (iv). There are signs that the GM(2,3) model achieves slightly more improvement in fit over the GM(1,3) model for females than for males. However, particularly as this greater improvement in fit is less significant in the later years of the data set, it is unlikely to be a sufficient reason to apply the GM(2,3) rather than the GM(1,3) model to England and Wales female data.

As an alternative visualisation of the structure of the standardised residuals, Figures 2.18-2.20 were projected onto the age axis and the year of birth diagonal to produce scatter diagrams. Figures 2.21 and 2.22 are the projections of the CMI standardised residuals onto the age axis and the year of birth diagonal respectively. Figures 2.23 and 2.24 are the projections of the England and Wales male standardised residuals onto the age axis and the year of birth diagonal respectively. Figures 2.25 and 2.26 are the projections of the England and Wales female standardised residuals onto the age axis and the year of birth diagonal respectively.

The following observations can be made from Figures 2.21-2.26:

- As we move from the simpler models to the more complex models, there are progressively fewer standardised residuals that are large in absolute value. This is to be expected because introducing each new time series of parameters captures more of the structure of the data.

- Figures 2.21(i) and (ii) show that, in the GM(0,2) and GM(1,2) models fitted to CMI data, the standardised residuals close to age 30 are mostly positive, those around age 40 are mostly negative, those around age 60 are mostly positive and those close to age 90 are mostly negative. This indicates that these models systematically understate mortality close to age 30, systematically overstate it around age 40, systematically understate it around age 60 and systematically overstate it close to age 90. This is consistent with what we observed from Figures 2.18(i) and (ii). As we move to the GM(1,3) model and to the more complex models, these systematic patterns in the sign of the residuals are largely eliminated.

- Even in the most complex models fitted to the CMI data, there is still evidence of the residuals being systematically positive in the age range 60-65 and negative in the age range 65-70. This feature was commented on in the discussion of Figures 2.18(iii) and (iv), where it was suggested that the explanation was that the exposed to risk at ages 60-65 consisted primarily of endowments, while that at ages 65-70 consisted primarily of whole life assurances. This feature is not observed in England and Wales data, which is what we would expect if this explanation is correct.

- For the CMI data, the graphs for the GM(3,3) and GM(2,4) models are very similar to those for the GM(2,3) model. For the England and Wales data, the graphs for more complex models than GM(2,3) have not been shown. Some differences between the graphs for the GM(1,3) and GM(2,3) models are visible for both CMI data and England and Wales data, but none of these differences is likely to be regarded as significant grounds for preferring one model to the other.

- In Figure 2.24, relating to England and Wales male data, and Figure 2.26, relating to England and Wales female data, the points relating to year of birth 1919 tend to have large negative standardised residuals and the points relating to year of birth 1920 tend to have large positive standardised residuals. As noted in the discussion of

Figures 2.19 and 2.20, this is likely to be due to the 1919 influenza epidemic. The same effect can be seen to some extent in CMI data in Figure 2.22, although it is less prominent.

- In England and Wales data compared with CMI data, there are more standardised residuals that are large in absolute value. As discussed in Section 2.6, this can be explained by the larger size of the England and Wales data sets than of the CMI data set, in that the England and Wales data are likely to contain more statistically significant features not captured by the models, and the more approximate procedure for calculating the exposed to risk for the England and Wales data than for the CMI data is also likely to have an effect.

- For the GM(0,2) model fitted to England and Wales male data, Figure 2.23(i) indicates that the residuals around age 70 are almost all positive, and that the residuals close to age 89 are almost all negative. This agrees with what we observed from Figure 2.19(i), namely a systematic understatement of mortality around age 70 and a systematic overstatement at ages close to 89. In both cases, it is necessary to introduce $\kappa^{(5)}$ parameters as well as $\kappa^{(0)}$ parameters to make the bias insignificant. Figure 2.23 also shows that the GM(0,2) and GM(1,2) models do not adequately capture the age structure of the England and Wales male data at younger ages, but that the GM(1,3) and GM(2,3) models give greatly improved results.

- Cohort effects can be seen under the simpler models in all three of the figures 2.22, 2.24 and 2.26, over the ranges of years of birth that we would expect from our earlier remarks. For CMI data, the fall in standardised residuals which can be seen in Figure 2.22(i), representing the favourable cohort effect centred on 1926, has largely been eliminated in the more complex models, indicating that most of this cohort effect has been captured implicitly via the shape of the kappa parameter graphs. Again for CMI data, Figures 2.22(i) and (to a lesser extent) (ii) show an increase in standardised residuals for lives born since 1945, but this has been captured implicitly via the kappa parameter estimates in the more complex models. Figure 2.24(i), for England and Wales male data, also shows an increase in standardised residuals for lives born since 1945, but this time introducing $\kappa^{(0)}$ parameters is sufficient to capture this effect implicitly. However, in the case of the favourable cohort effect centred on 1931 for both England and Wales males and England and Wales females, and the adverse cohort effect centred on 1915 for England and Wales females, many of the cohort effects remain in the residuals of the more complex models. For England and Wales males, the significant fall in the standardised residuals centred on 1931 visible in Figures 2.24(i) and (ii) has been noticeably reduced, but by no means eliminated, in Figures 2.24(iii) and (iv). For England and Wales females, the GM(1,3) and GM(2,3) models are still less successful at capturing the fall in the standardised residuals centred on 1931 visible under the GM(0,2) and GM(1,2) models. Also for England and Wales females, the GM(1,3) and GM(2,3) models have only limited success at capturing the rise in standardised residuals from 1905 to 1925, which is interrupted by the 1919-20 discontinuity.

- From Figure 2.25(i), the residuals when the GM(0,2) model is fitted to England and Wales female data are mostly positive up to age 50 and above age 80 and negative around age 70. This indicates systematic understatement of mortality up to age 50 and above age 80 and systematic overstatement around age 70, consistently with Figure 2.20(i). From Figure 2.25(ii), when $\kappa^{(0)}$ parameters are introduced, the residuals close to age 30 become mostly negative, those around ages 50 and 70 mostly remain positive and negative respectively and those close to age 89 become negative at certain years (actually in earlier years, although this cannot be seen from Figure 2.25(ii)). This agrees with Figure 2.20(ii).

- In the GM(1,3) and GM(2,3) models fitted to England and Wales female data (Figures 2.25(iii) and (iv)) rather than male data (Figures 2.23(iii) and (iv)), there is rather more age structure remaining in the residuals. However, this should not be taken as support for the use of a more complex GM model than the GM(2,3) model for

England and Wales female data. We observed in Section 2.3 that even in the GM(2,3) model, there were significant problems of overfitting.

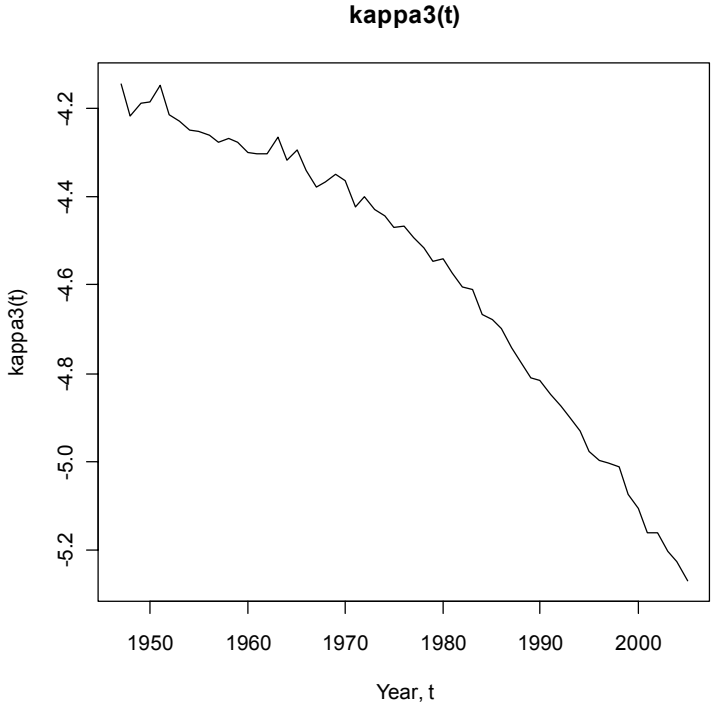
- Some negative standardised residuals can be seen at the oldest ages in all the graphs in Figure 2.25, relating to England and Wales female data. It seems likely that these are due to issues with the quality of the data, with ages at death in the late 80s that were not known exactly being reported as 90 which is outside the range of the graphs. The graphs in Figure 2.20 suggest that these negative standardised residuals occur mainly in the earlier years of the data set, so that this issue has become less significant in more recent years.

2.9 Conclusion

A number of different GM models have been fitted to the three data sets. It was found that the parameter graphs for the GM(1,3) model had a clear interpretation but that the graphs for more complex GM models were much more difficult to interpret. In addition, no major features of the data have been identified that were not captured in the GM(1,3) model but were in more complex GM models. Therefore, despite the fact that some more complex GM models improved the AIC and BIC compared with the GM(1,3) model, we conclude that the GM(1,3) model is the most appropriate model on which to base future mortality projections. Cohort effects are reflected indirectly via the fitted parameters of this model, but there are still some cohort effects visible in the residuals, particularly for the England and Wales data sets. We shall next extend the model to incorporate an explicit allowance for these cohort effects.

Figure 2.1 – Maximum likelihood parameter estimates for the GM(0,2) model fitted to CMI data **–** $\mu_{xt} = \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x})]$ – (i) $\kappa_t^{(3)}$, (ii) $\kappa_t^{(4)}$

(i)



(ii)

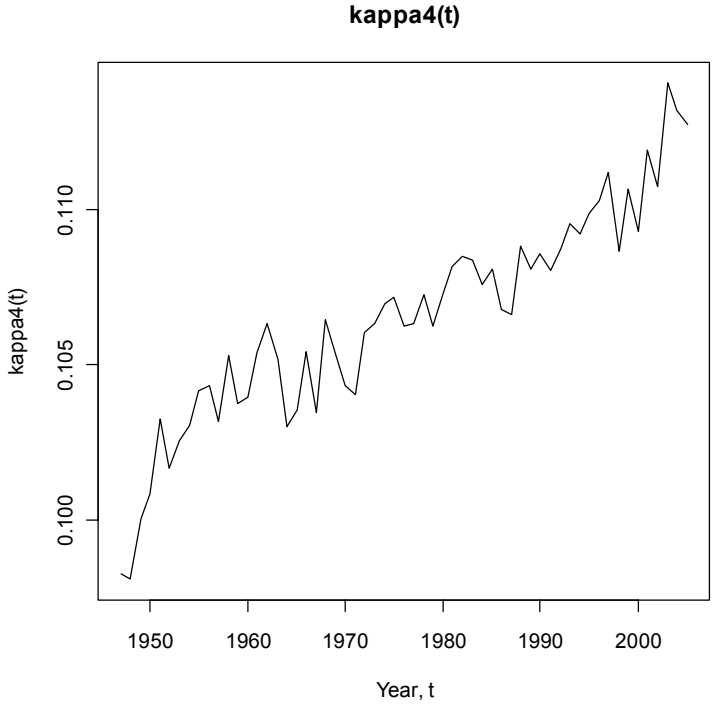
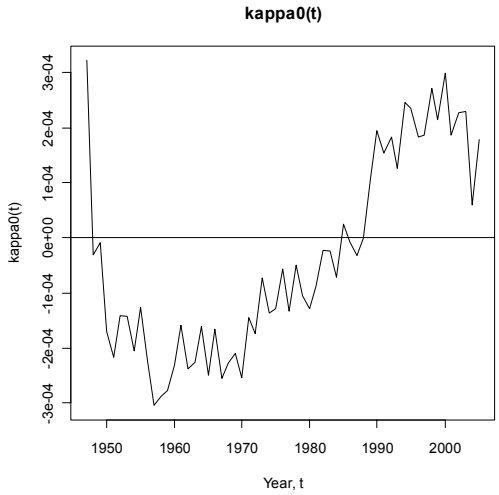
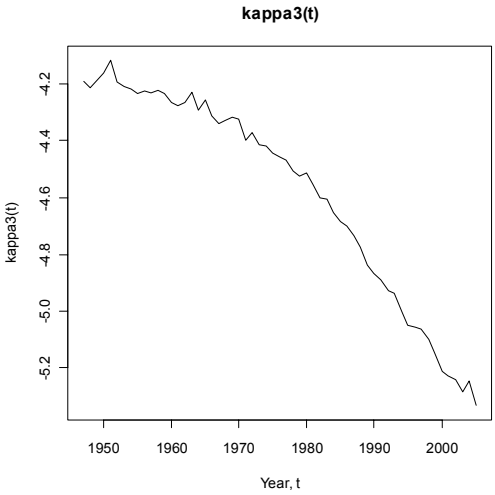


Figure 2.2 – Maximum likelihood parameter estimates for the GM(1,2) model fitted to CMI data **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x})]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$

(i)



(ii)



(iii)

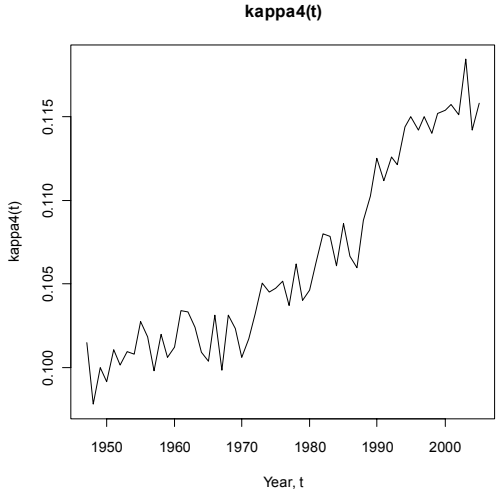
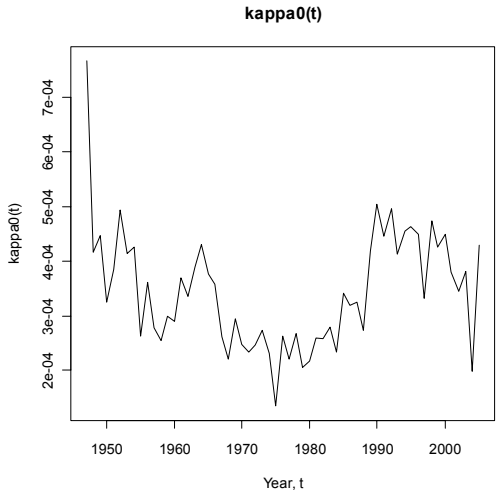
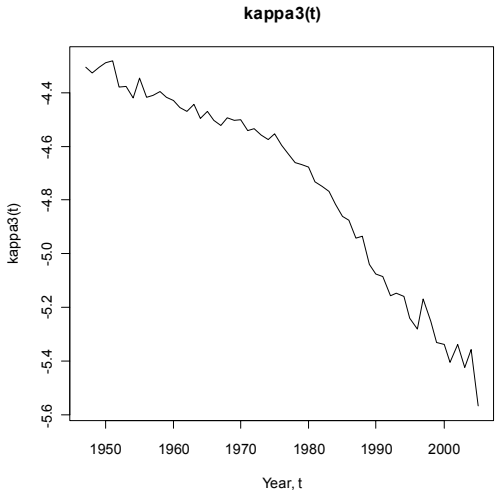


Figure 2.3 – Maximum likelihood parameter estimates for the GM(1,3) model fitted to CMI data **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x}) + \kappa_t^{(5)}((x - \bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$

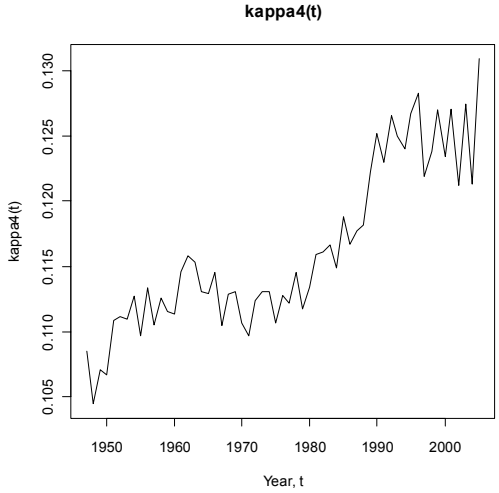
(i)



(ii)



(iii)



(iv)

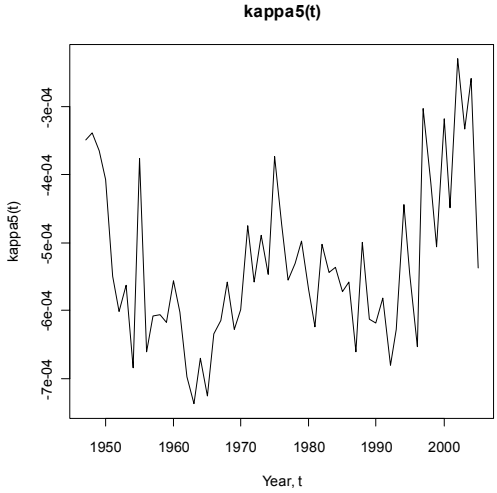
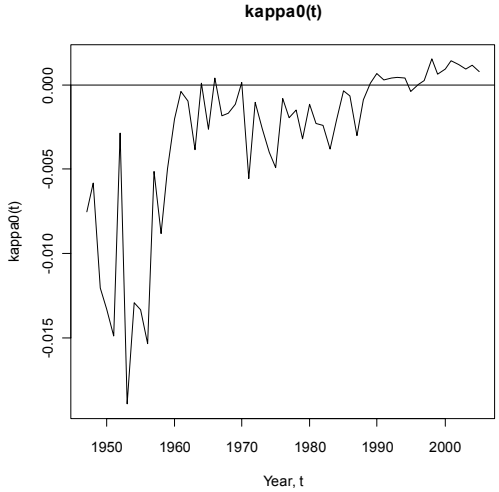
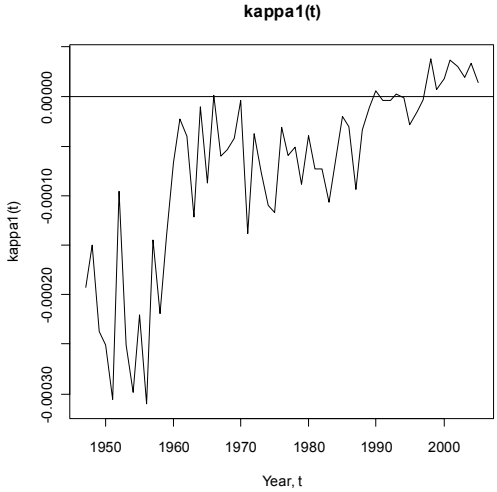


Figure 2.4 – Maximum likelihood parameter estimates for the GM(2,3) model fitted to CMI data **–** $\mu_{xt} = \kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(1)}$, (iii) $\kappa_t^{(3)}$, (iv) $\kappa_t^{(4)}$, (v) $\kappa_t^{(5)}$

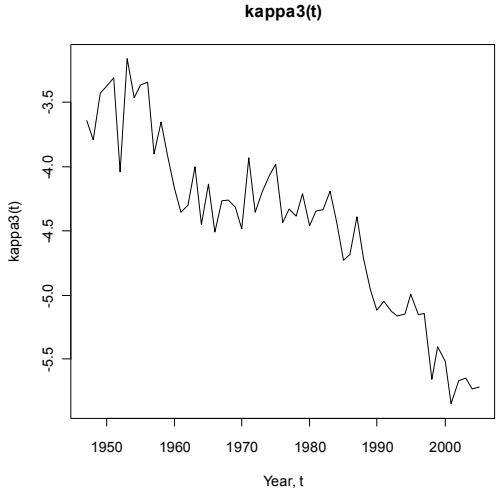
(i)



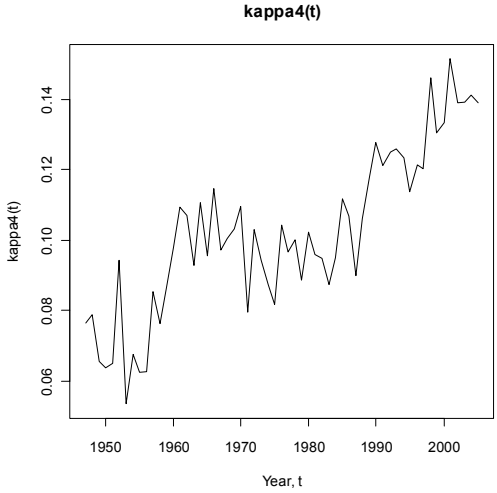
(ii)



(iii)



(iv)



(v)

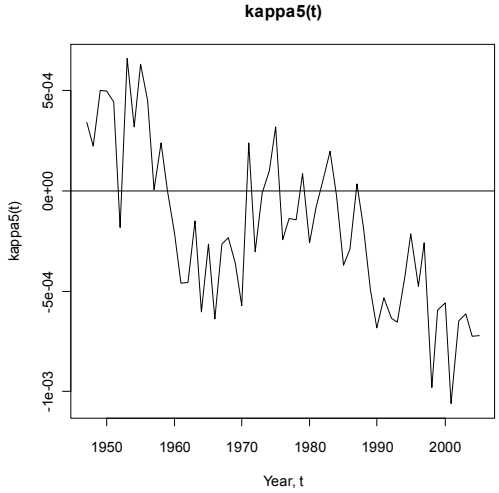
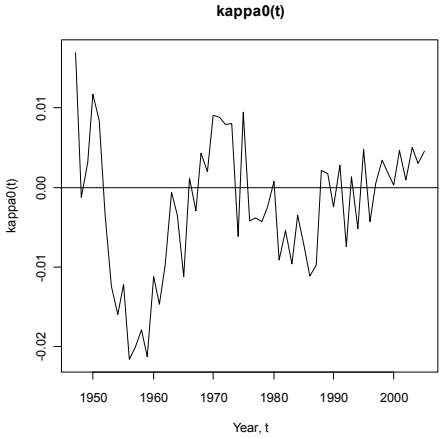
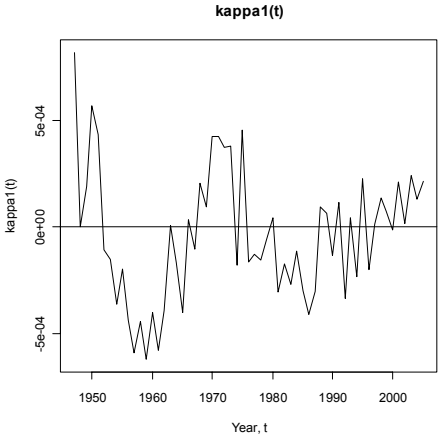


Figure 2.5 – Maximum likelihood parameter estimates for the GM(3,3) model fitted to CMI data – $\mu_{xt} = \kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \kappa_t^{(2)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]$ – (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(1)}$, (iii) $\kappa_t^{(2)}$, (iv) $\kappa_t^{(3)}$, (v) $\kappa_t^{(4)}$, (vi) $\kappa_t^{(5)}$

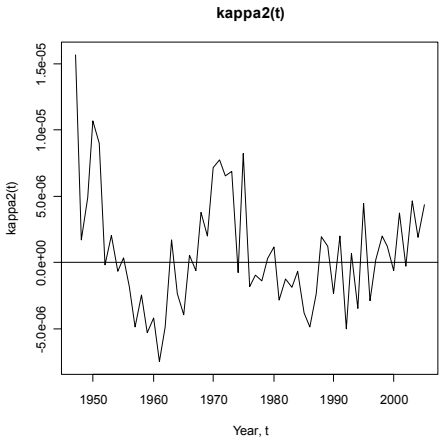
(i)



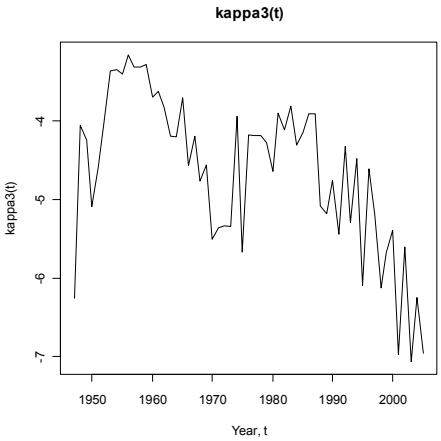
(ii)



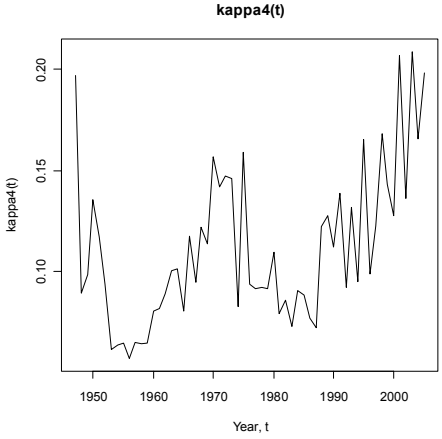
(iii)



(iv)



(v)



(vi)

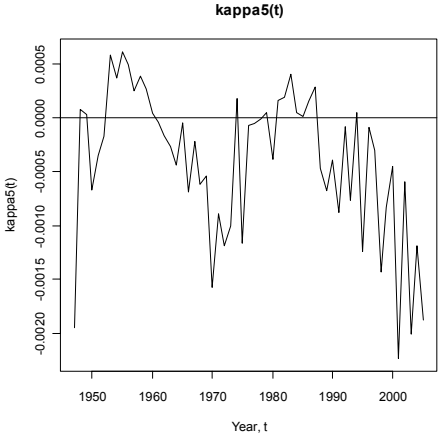
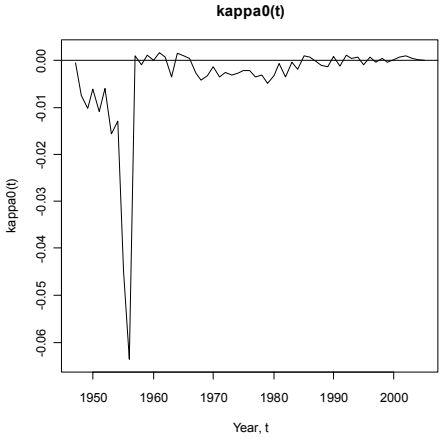
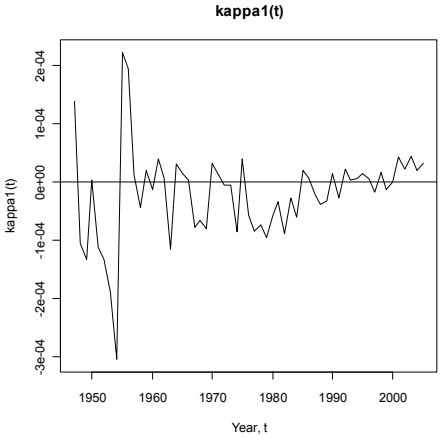


Figure 2.6 – Maximum likelihood parameter estimates for the GM(2,4) model fitted to CMI data – $\mu_{xt} = \kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \kappa_t^{(6)}(x-\bar{x})^3]$ – (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(1)}$, (iii) $\kappa_t^{(3)}$, (iv) $\kappa_t^{(4)}$, (v) $\kappa_t^{(5)}$, (vi) $\kappa_t^{(6)}$

(i)



(ii)



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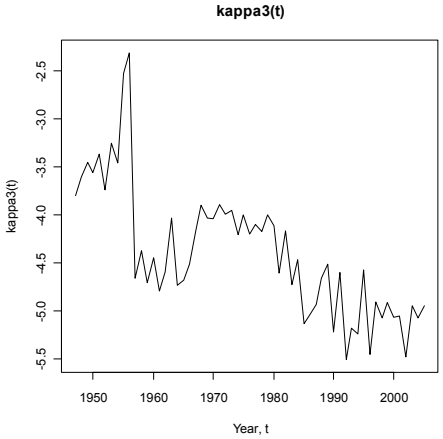

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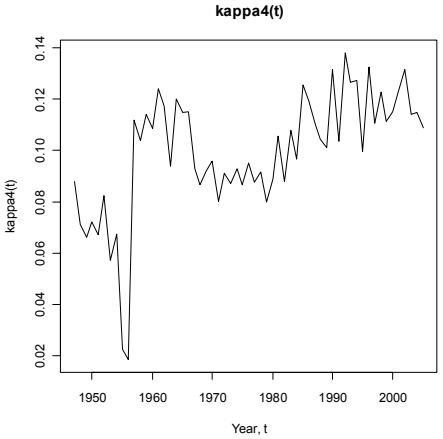

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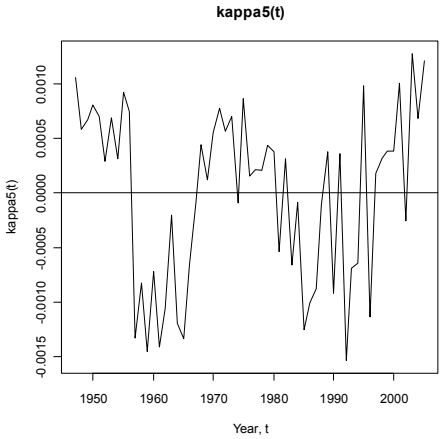


(vi)

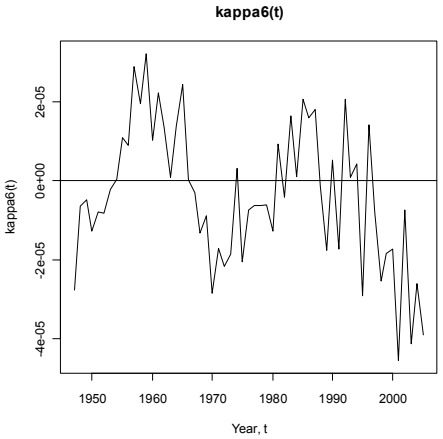
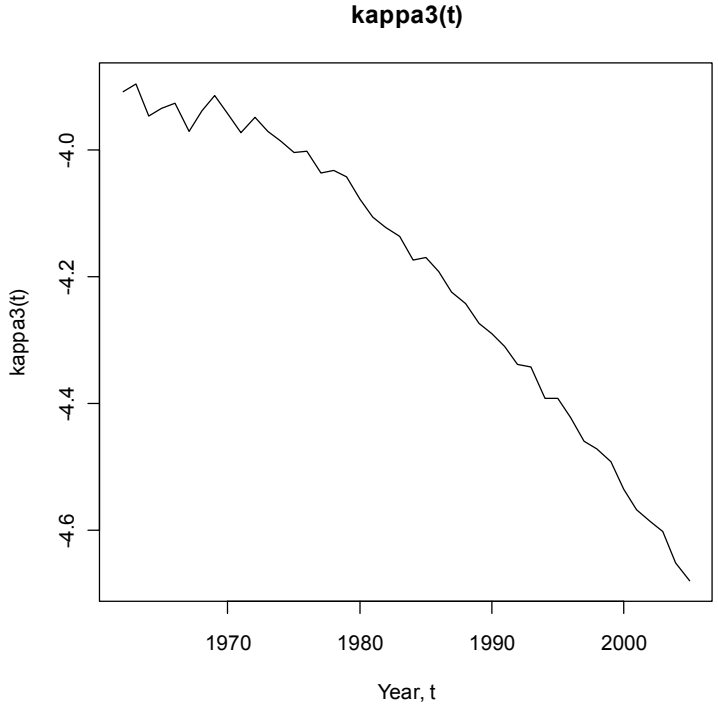


Figure 2.7 – Maximum likelihood parameter estimates for the GM(0,2) model fitted to England and Wales male data **–** $\mu_{xt} = \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x})]$ **–** (i) $\kappa_t^{(3)}$, (ii) $\kappa_t^{(4)}$

(i)



(ii)

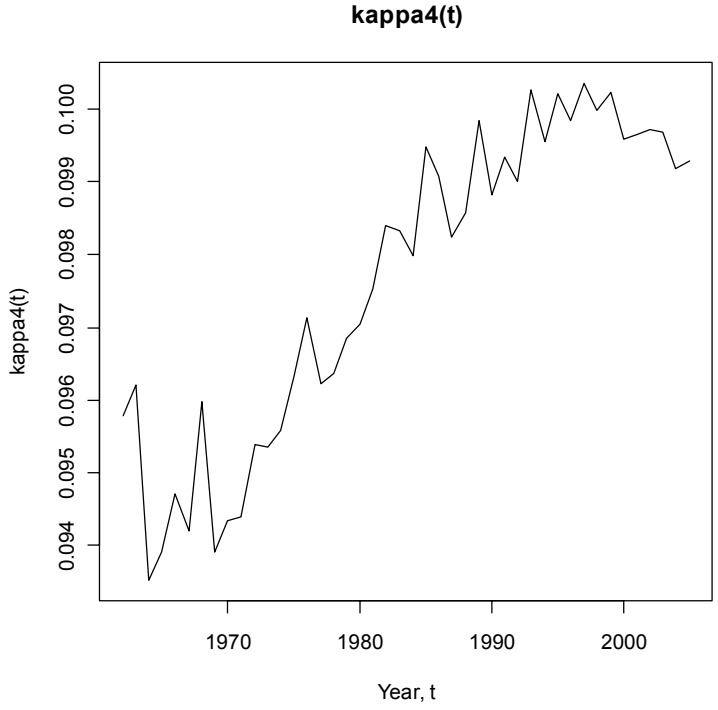
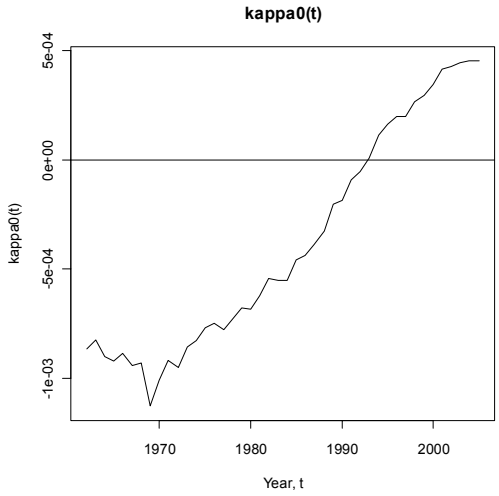
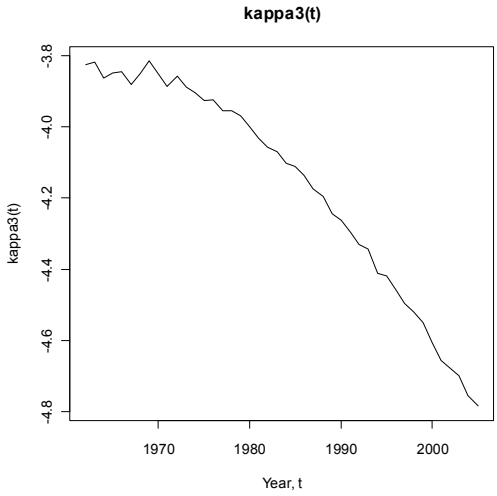


Figure 2.8 – Maximum likelihood parameter estimates for the GM(1,2) model fitted to England and Wales male data **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x})]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$

(i)



(ii)



(iii)

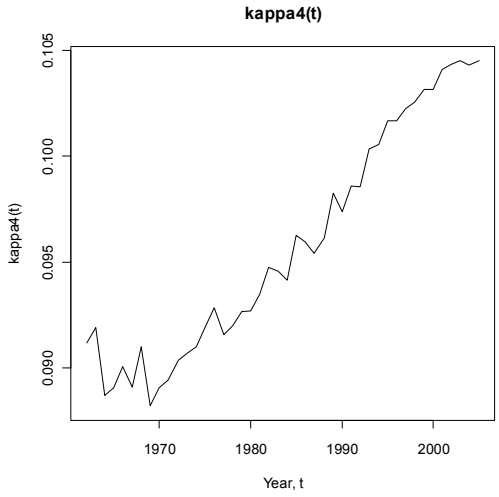
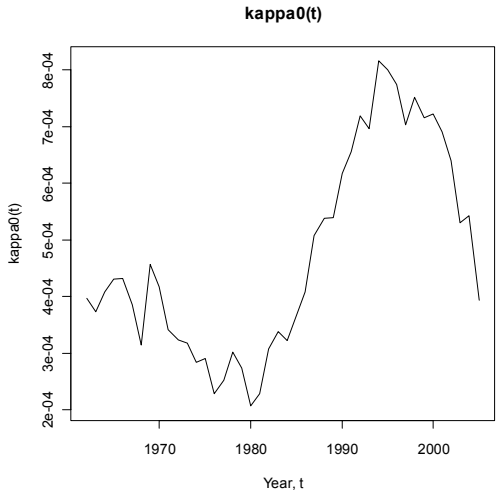
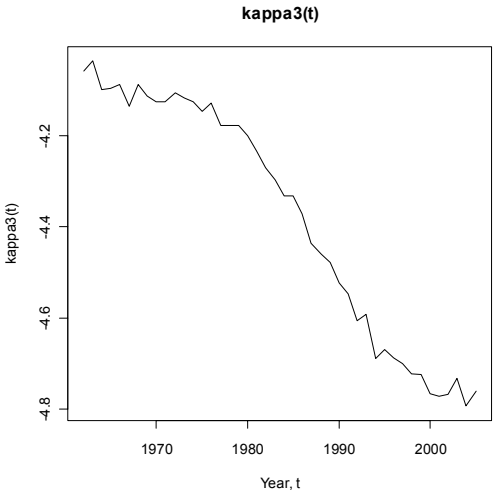


Figure 2.9 – Maximum likelihood parameter estimates for the GM(1,3) model fitted to England and Wales male data **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$

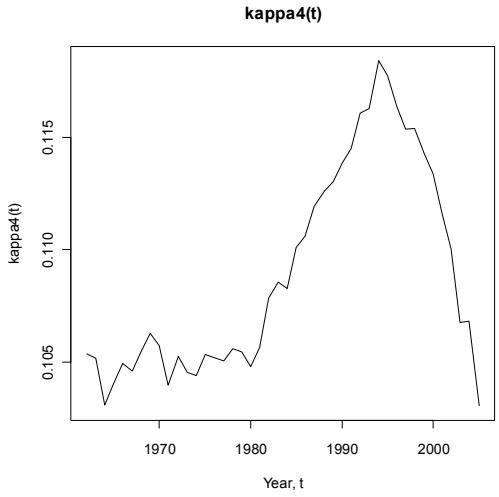
(i)



(ii)



(iii)



(iv)

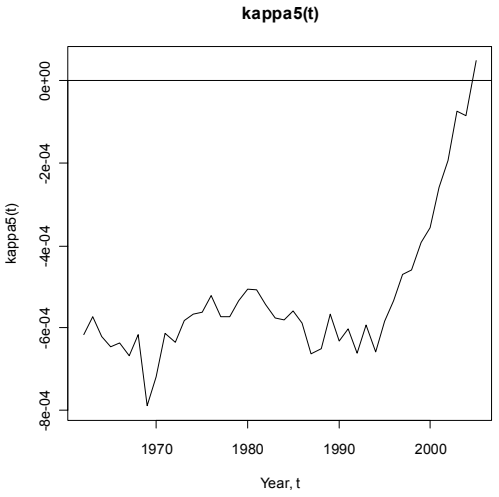
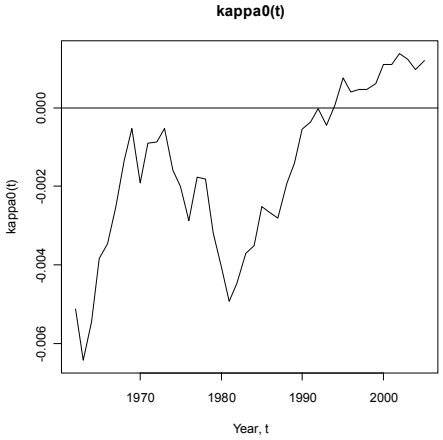


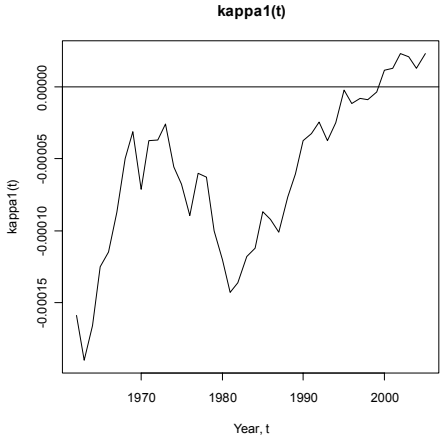
Figure 2.10 – Maximum likelihood parameter estimates for the GM(2,3) model fitted to England and Wales male data **–**

$\mu_{xt} = \kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(1)}$, (iii) $\kappa_t^{(3)}$, (iv) $\kappa_t^{(4)}$, (v) $\kappa_t^{(5)}$

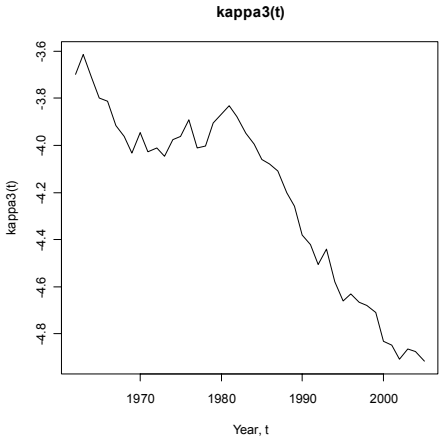
(i)



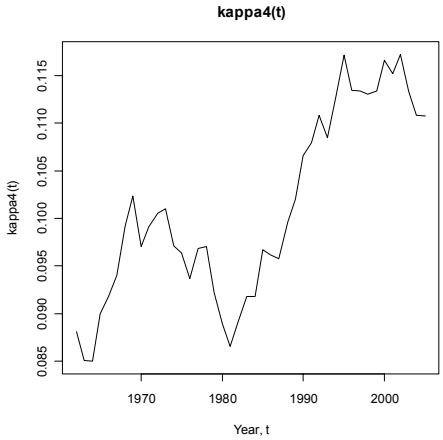
(ii)



(iii)



(iv)



(v)

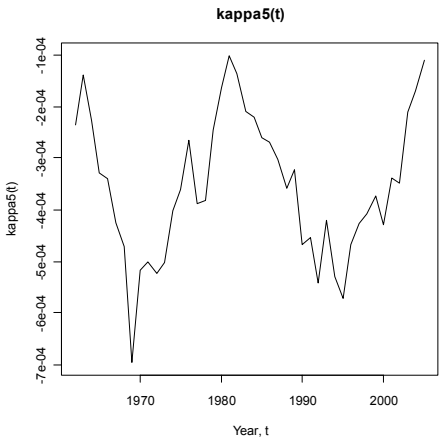
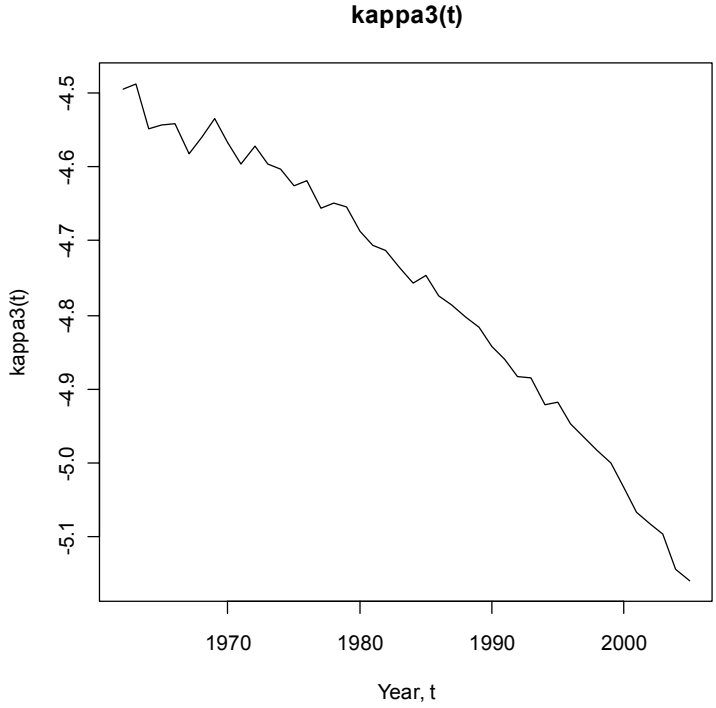


Figure 2.11 – Maximum likelihood parameter estimates for the GM(0,2) model fitted to England and Wales female data **–** $\mu_{xt} = \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x})]$ **–** (i) $\kappa_t^{(3)}$, (ii) $\kappa_t^{(4)}$

(i)



(ii)

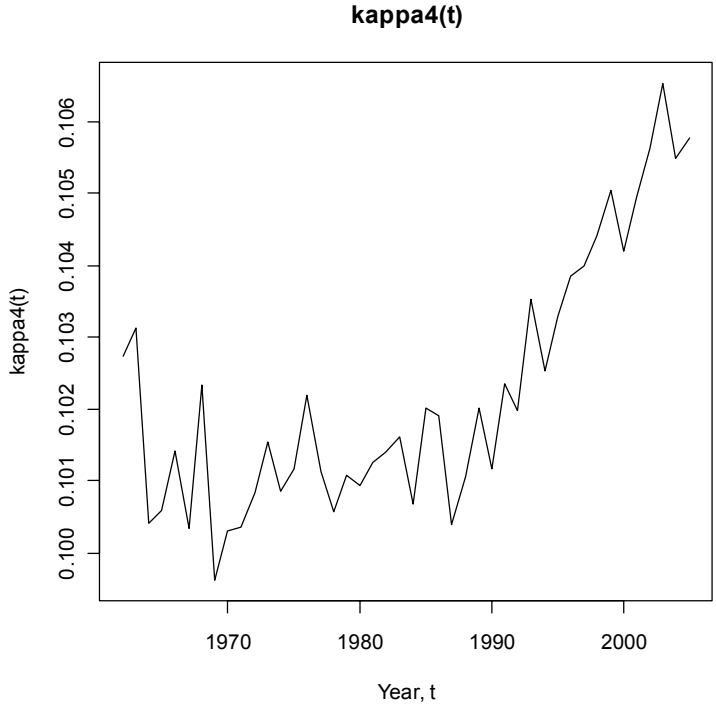
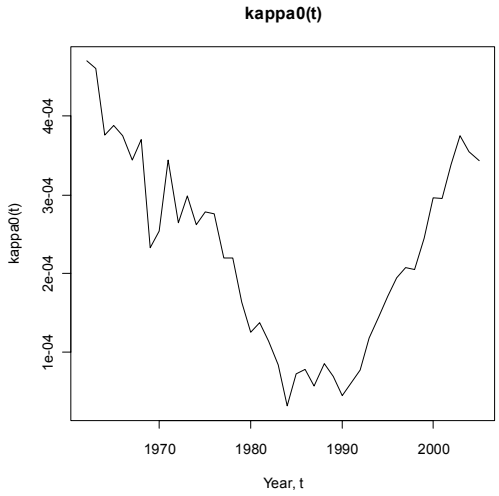
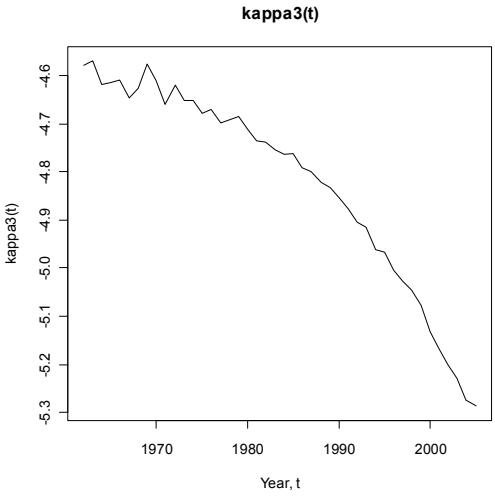


Figure 2.12 – Maximum likelihood parameter estimates for the GM(1,2) model fitted to England and Wales female data **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x})]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$

(i)



(ii)



(iii)

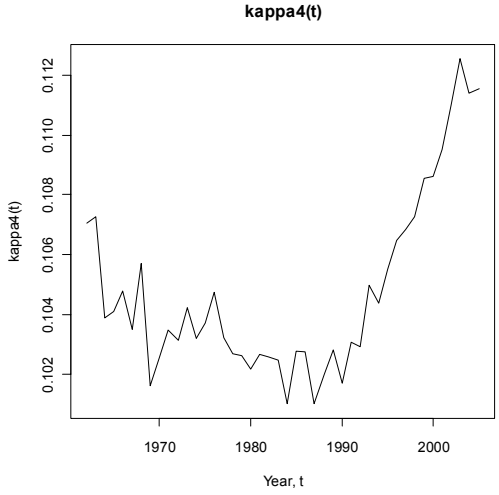
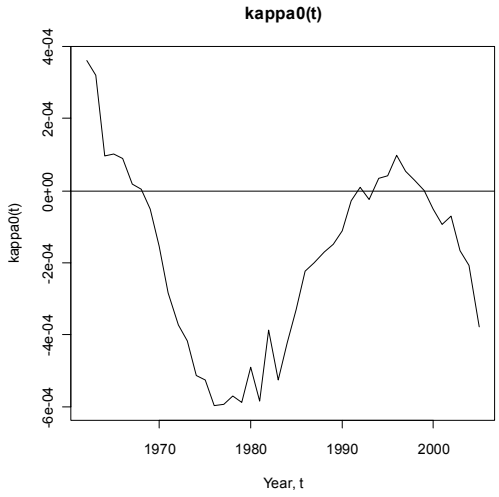
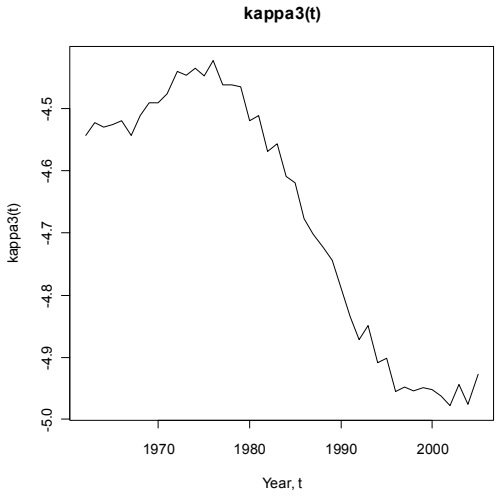


Figure 2.13 – Maximum likelihood parameter estimates for the GM(1,3) model fitted to England and Wales female data **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x}) + \kappa_t^{(5)}((x - \bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$

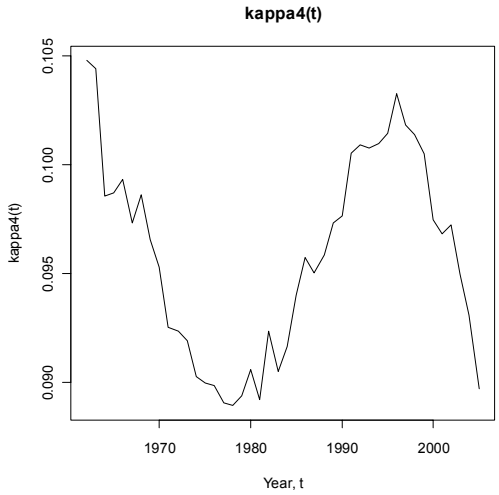
(i)



(ii)



(iii)



(iv)

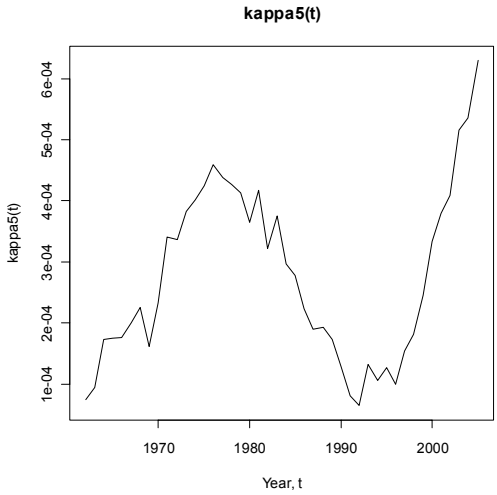
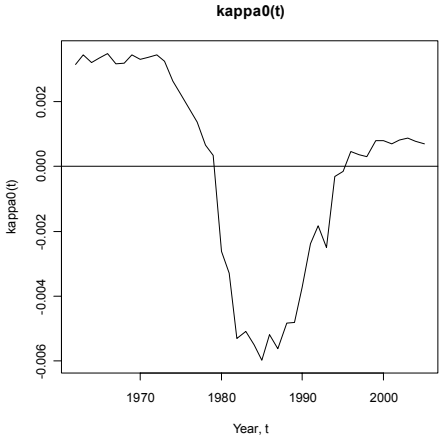
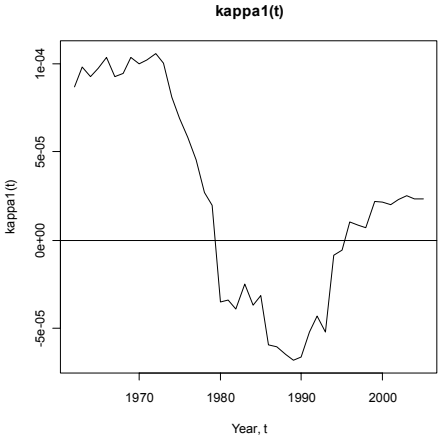


Figure 2.14 – Maximum likelihood parameter estimates for the GM(2,3) model fitted to England and Wales female data **–** $\mu_{xt} = \kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(1)}$, (iii) $\kappa_t^{(3)}$, (iv) $\kappa_t^{(4)}$, (v) $\kappa_t^{(5)}$

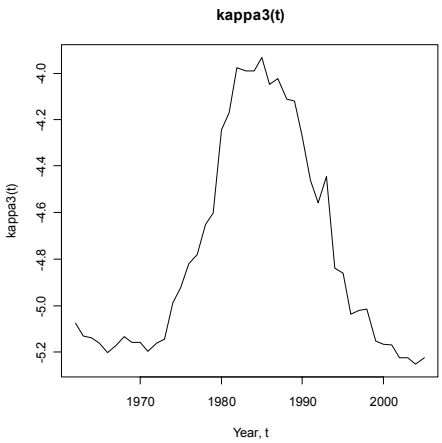
(i)



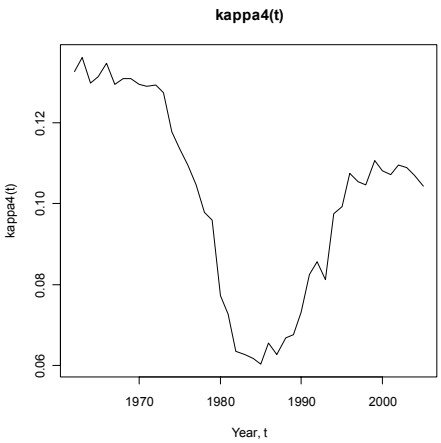
(ii)



(iii)



(iv)



(v)

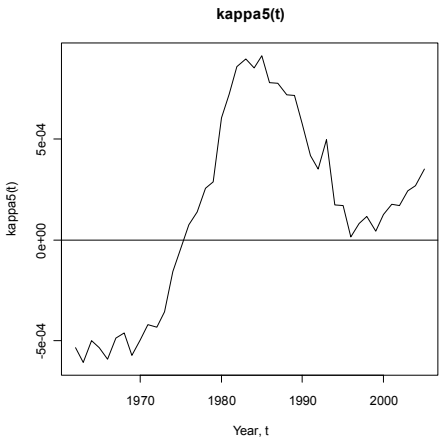
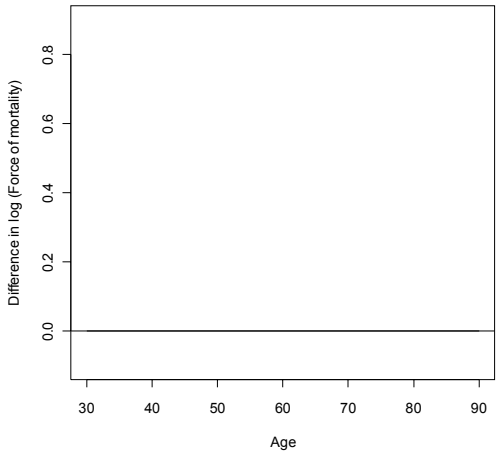
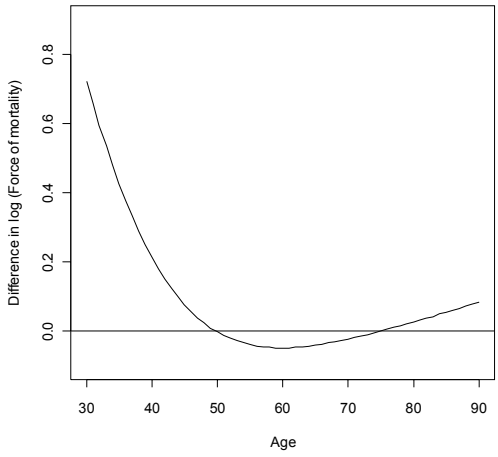


Figure 2.15 – 2000 mortality curves fitted to CMI data **–** differences from the GM(0,2) model **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3), (v) GM(3,3), (vi) GM(2,4)

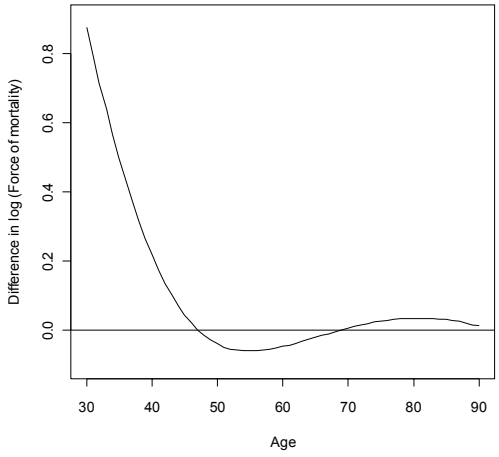
(i)



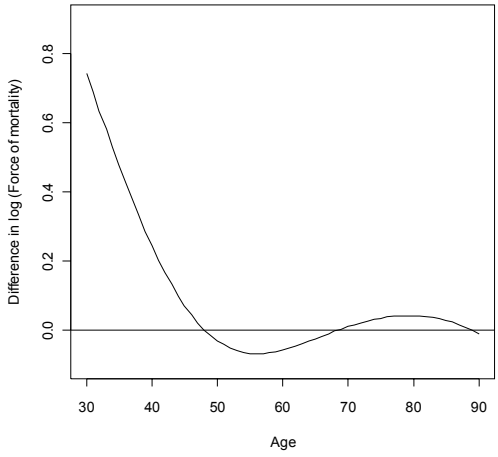
(ii)



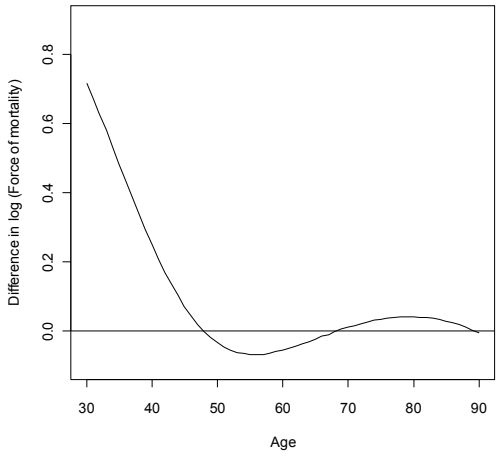
(iii)



(iv)



(v)



(vi)

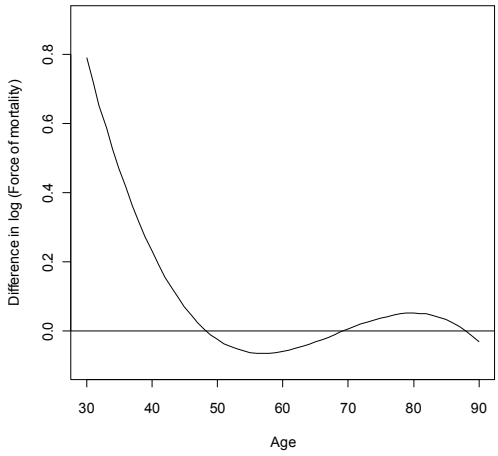
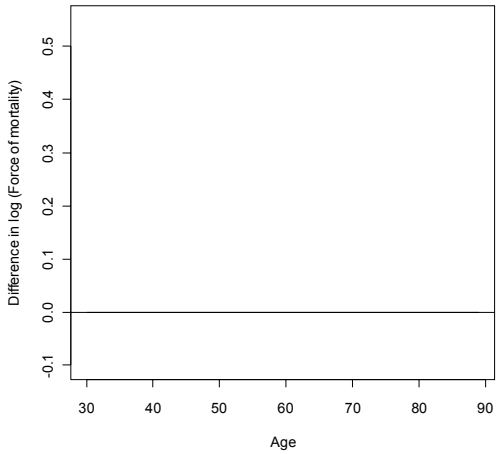
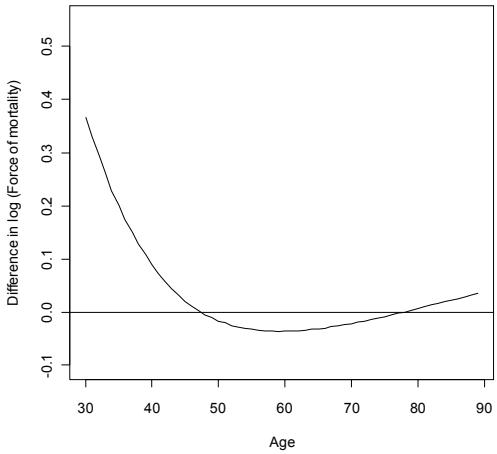


Figure 2.16 – 2000 mortality curves fitted to England and Wales male data – differences from the GM(0,2) model **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

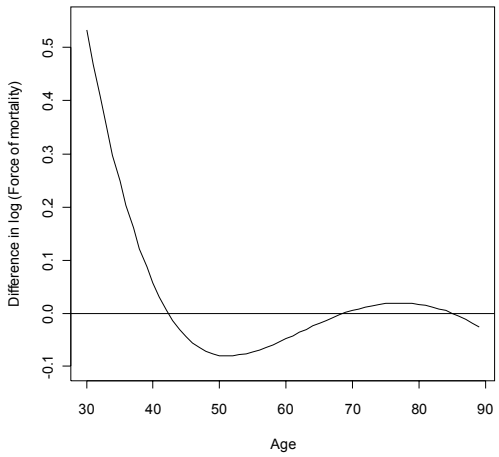
(i)



(ii)



(iii)



(iv)

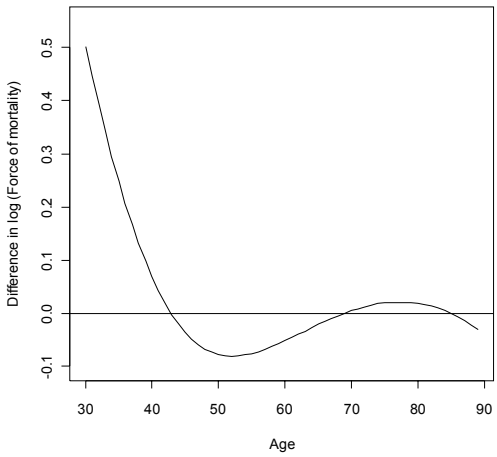
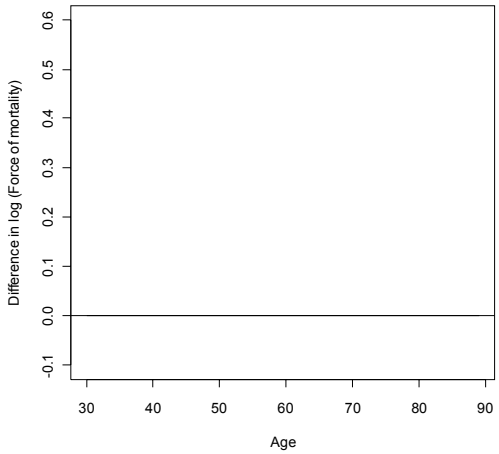
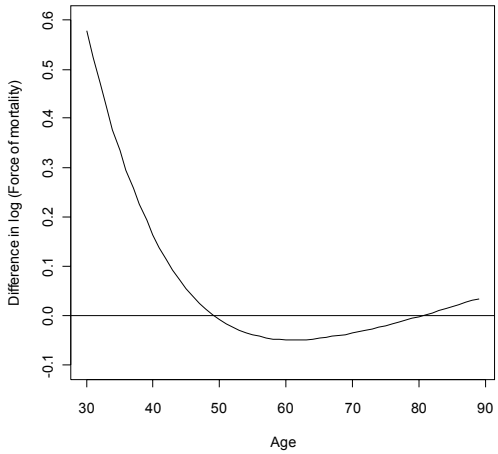


Figure 2.17 – 2000 mortality curves fitted to England and Wales female data **–** differences from the GM(0,2) model **–**(i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

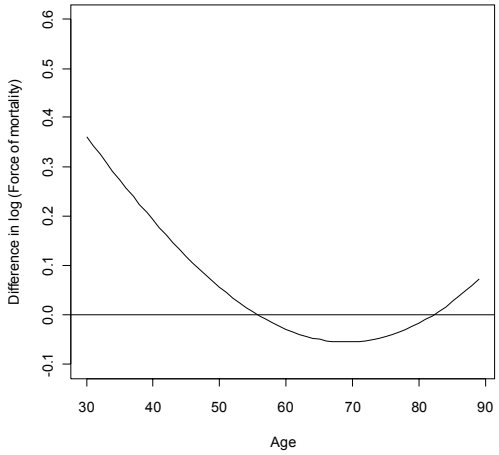
(i)



(ii)



(iii)



(iv)

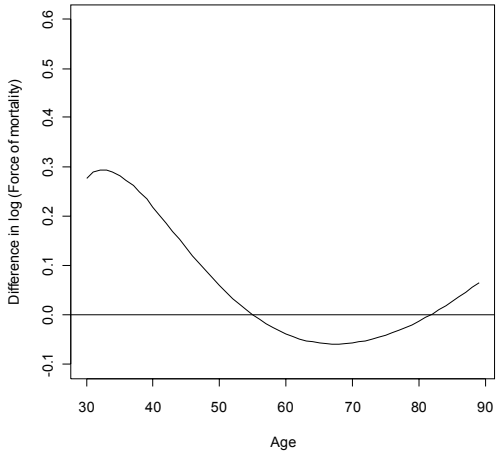
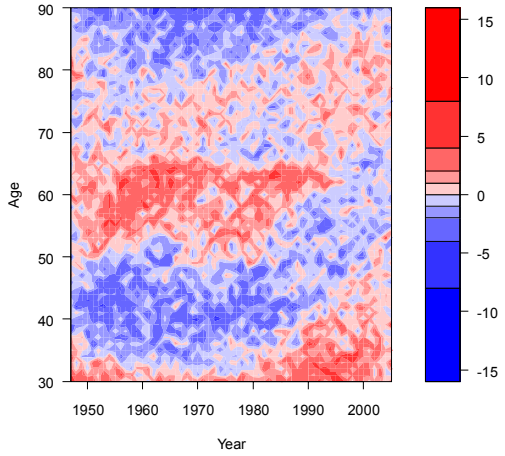
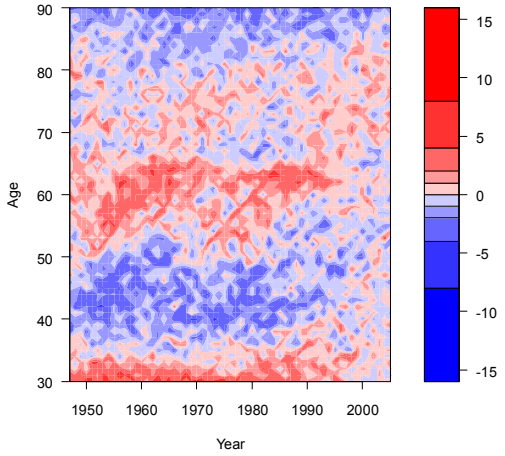


Figure 2.18 – Plots of standardised residuals for a number of possible GM models fitted to CMI data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3), (v) GM(3,3), (vi) GM(2,4)

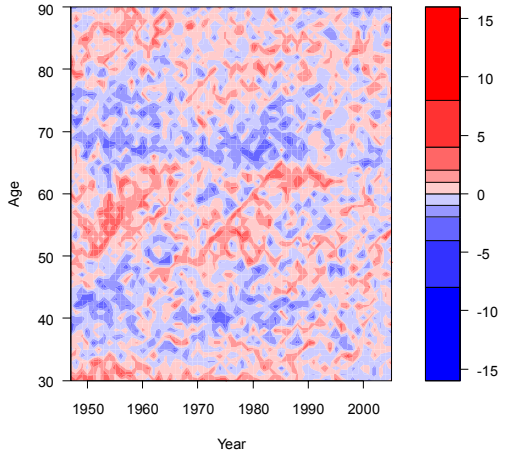
(i)



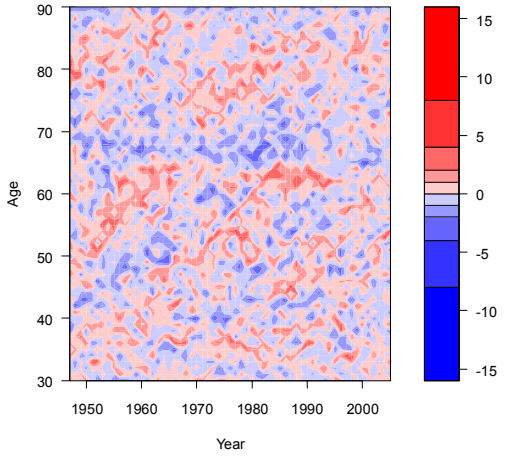
(ii)



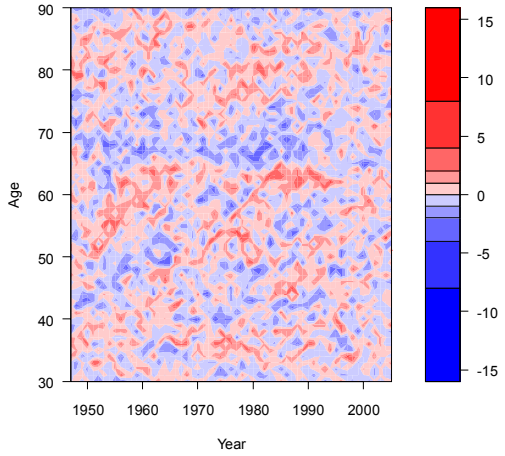
(iii)



(iv)



(v)



(vi)

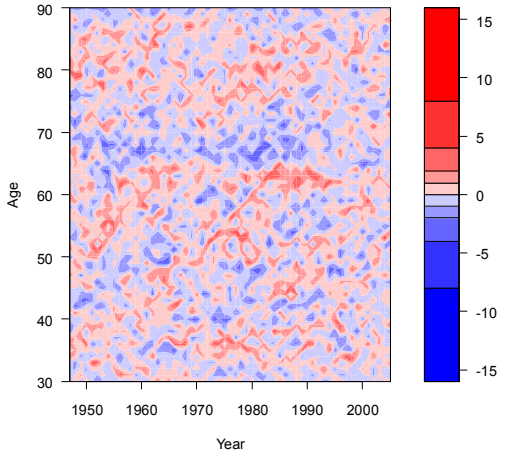
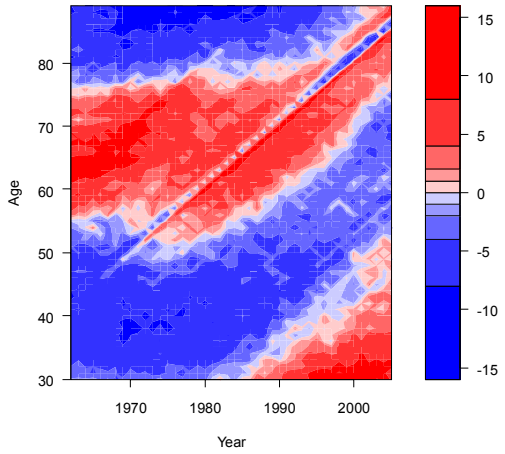
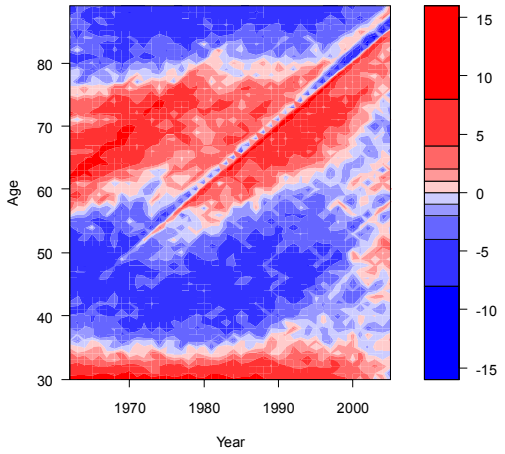


Figure 2.19 – Plots of standardised residuals for a number of possible GM models fitted to England and Wales male data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

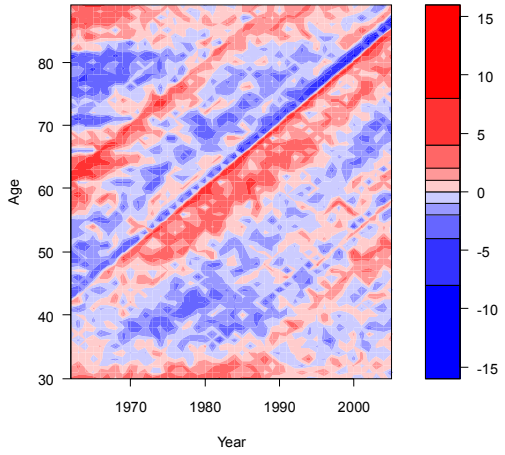
(i)



(ii)



(iii)



(iv)

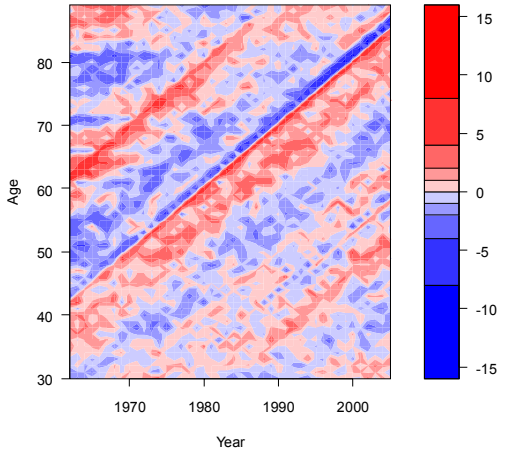
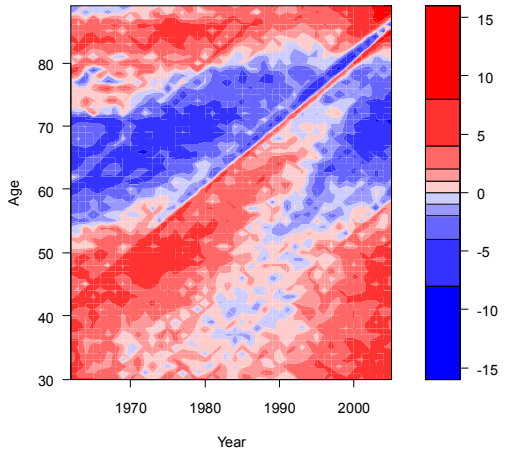
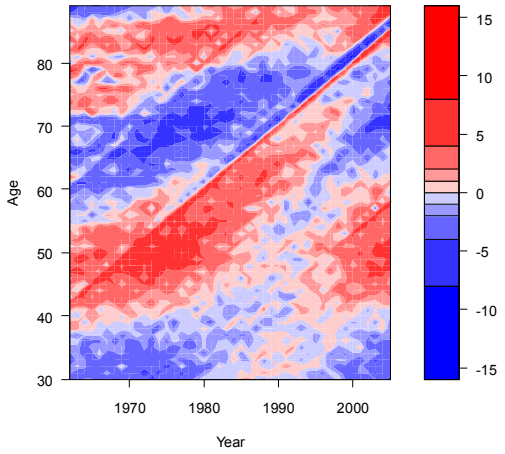


Figure 2.20 – Plots of standardised residuals for a number of possible GM models fitted to England and Wales female data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

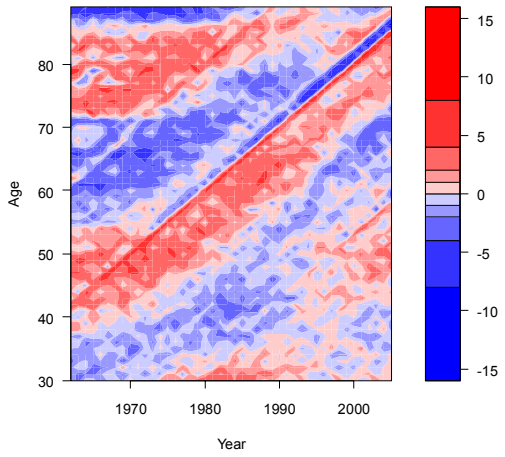
(i)



(ii)



(iii)



(iv)

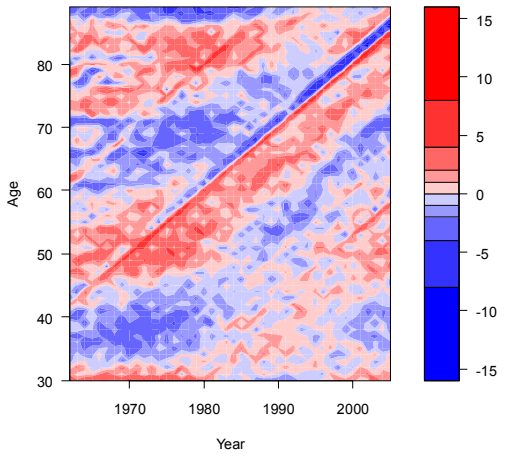
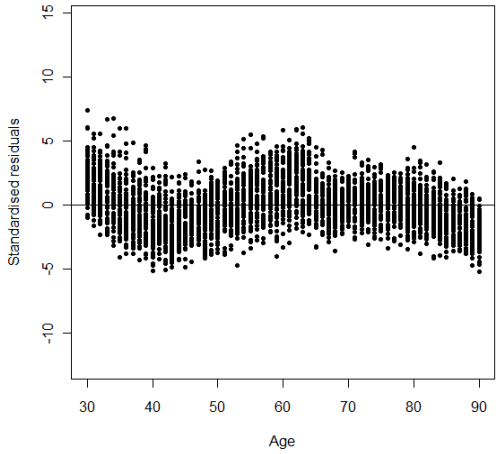
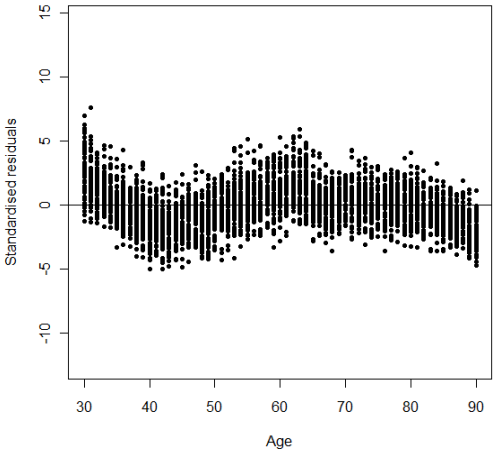


Figure 2.21 – Scatter diagrams of standardised residuals plotted against age for CMI data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3), (v) GM(3,3), (vi) GM(2,4)

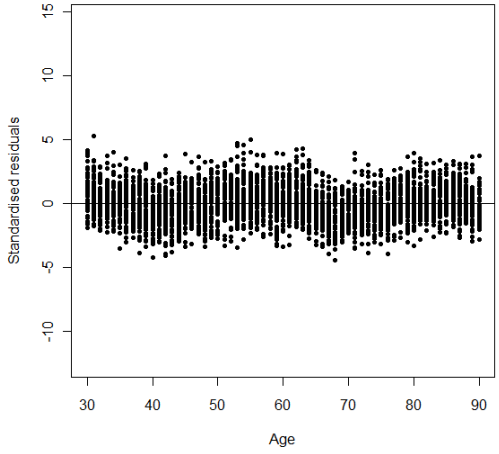
(i)



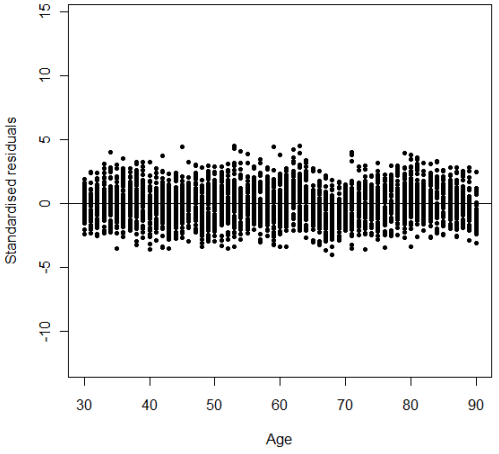
(ii)



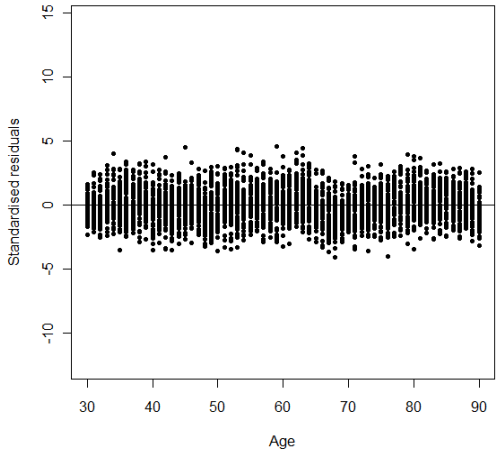
(iii)



(iv)



(v)



(vi)

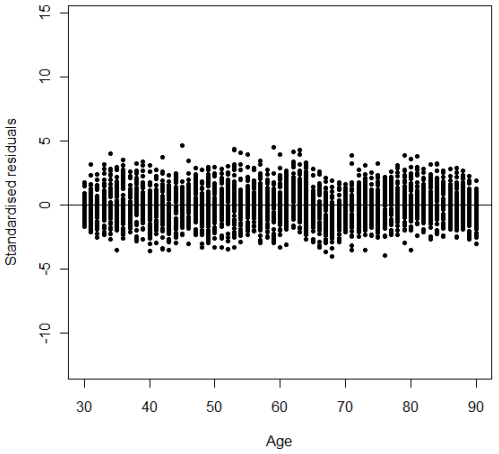
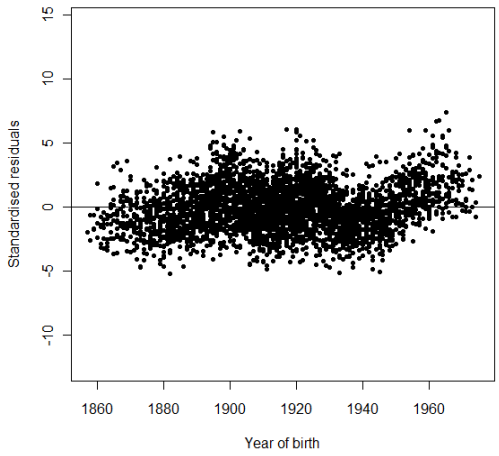
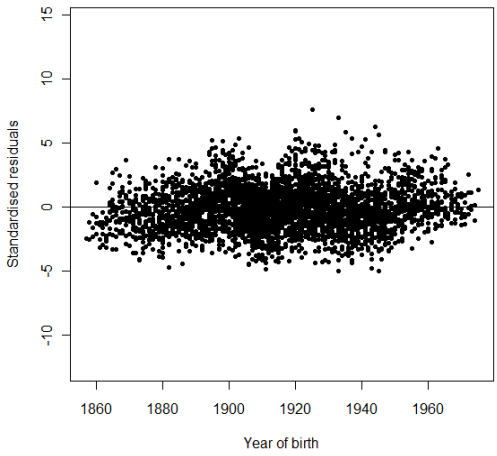


Figure 2.22 – Scatter diagrams of standardised residuals plotted against year of birth for CMI data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3), (v) GM(3,3), (vi) GM(2,4)

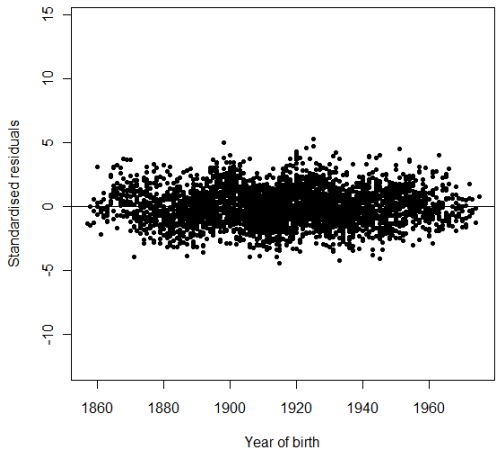
(i)



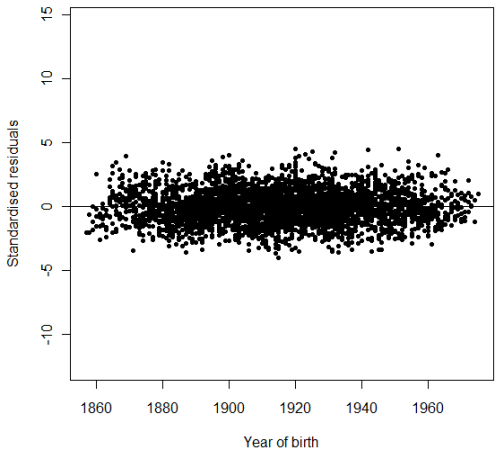
(ii)



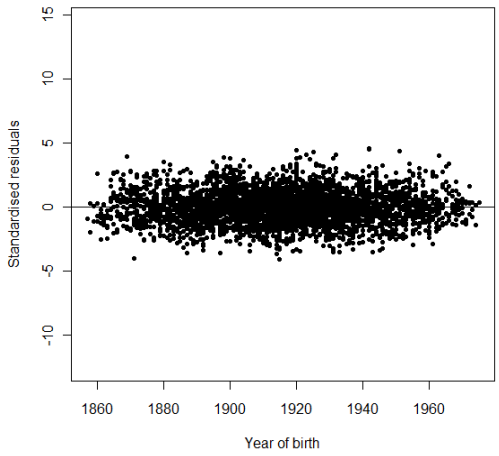
(iii)



(iv)



(v)



(vi)

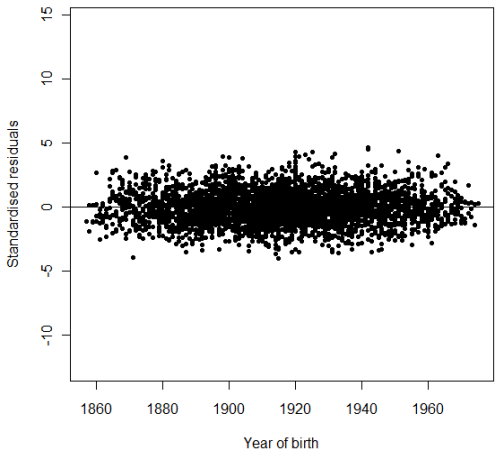
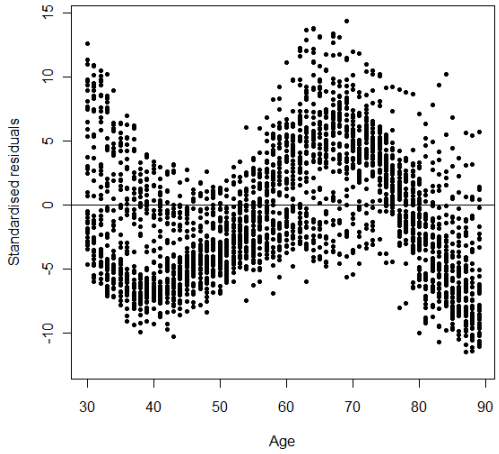
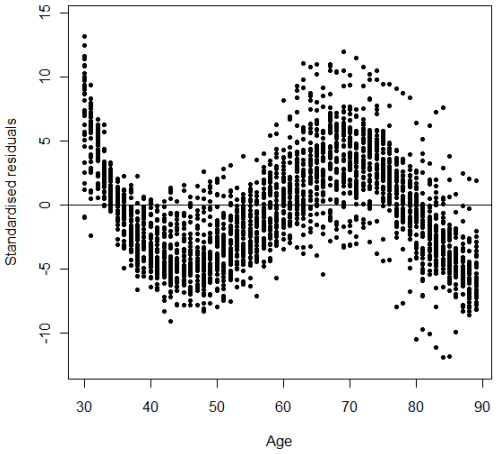


Figure 2.23 – Scatter diagrams of standardised residuals plotted against age for England and Wales male data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

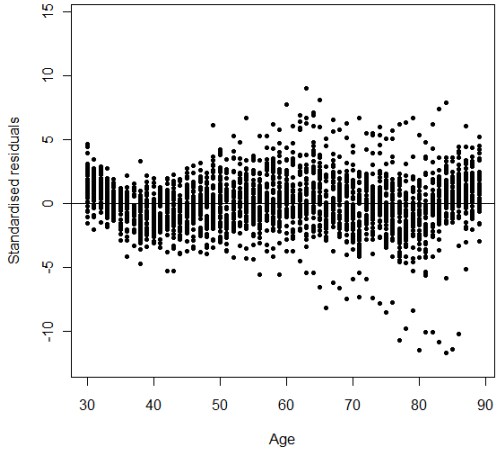
(i)



(ii)



(iii)



(iv)

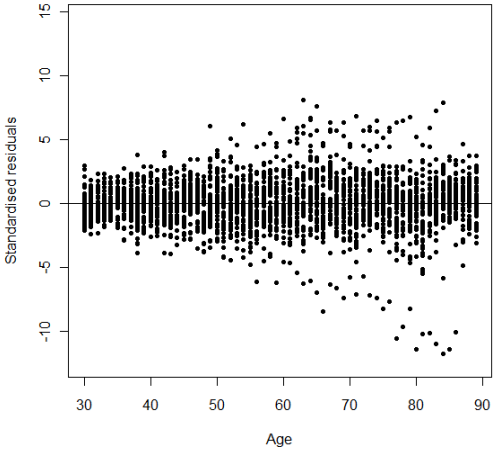
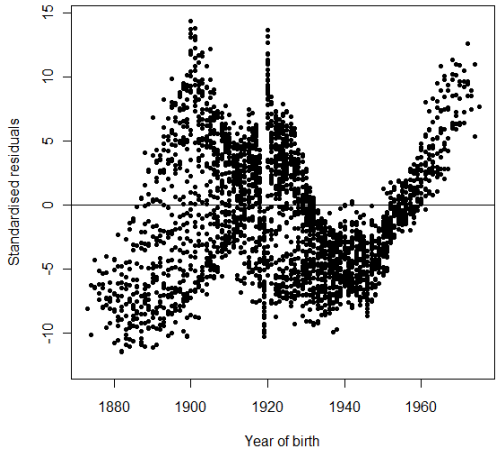
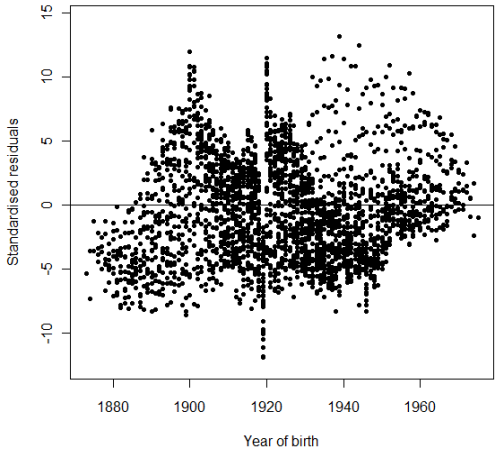


Figure 2.24 – Scatter diagrams of standardised residuals plotted against year of birth for England and Wales male data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

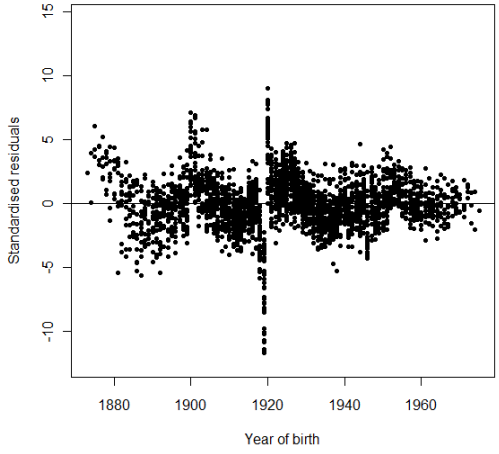
(i)



(ii)



(iii)



(iv)

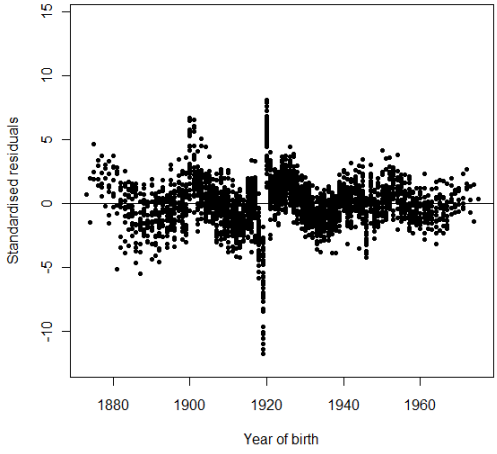
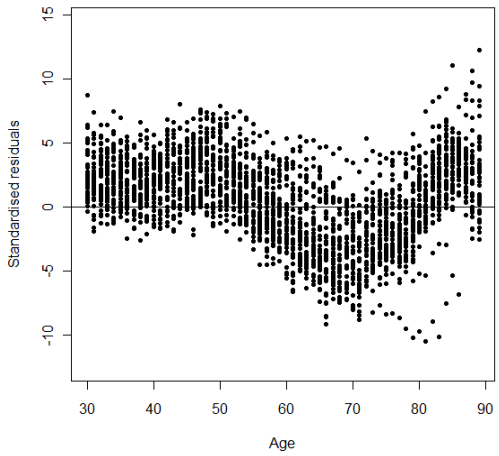
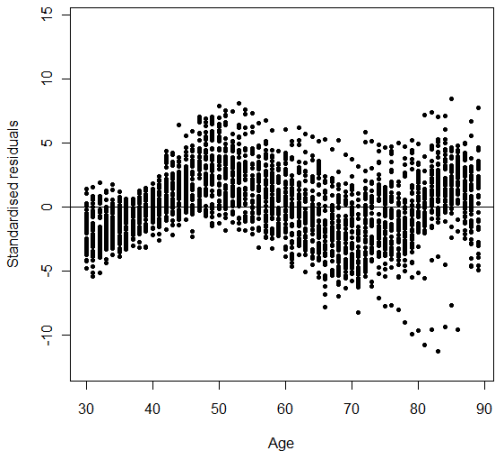


Figure 2.25 – Scatter diagrams of standardised residuals plotted against age for England and Wales female data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

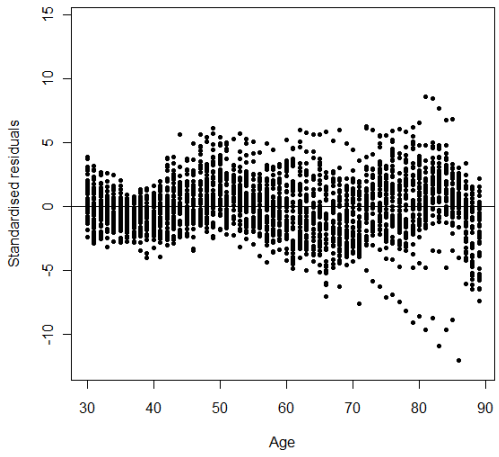
(i)



(ii)



(iii)



(iv)

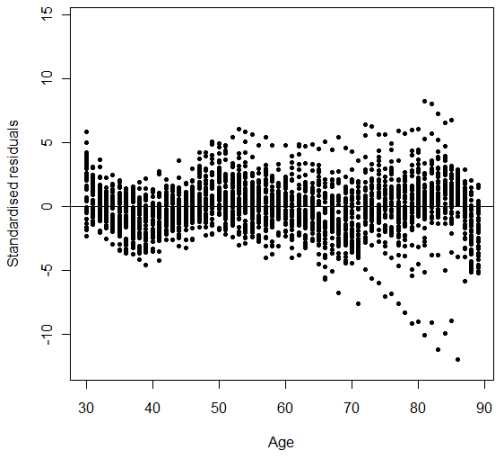
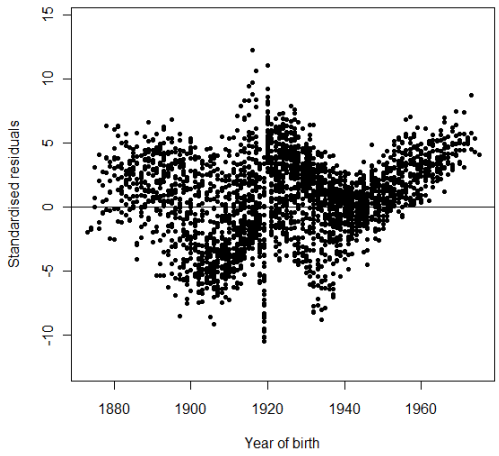
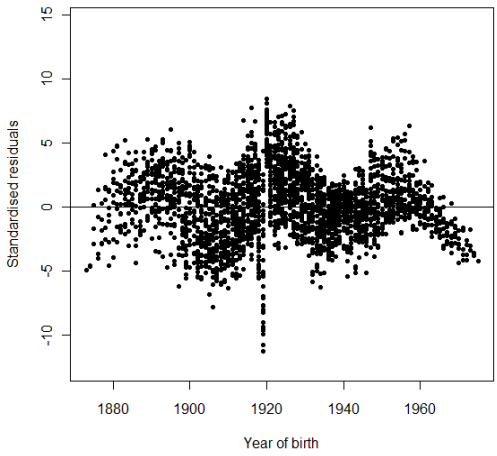


Figure 2.26 – Scatter diagrams of standardised residuals plotted against year of birth for England and Wales female data **–** (i) GM(0,2), (ii) GM(1,2), (iii) GM(1,3), (iv) GM(2,3)

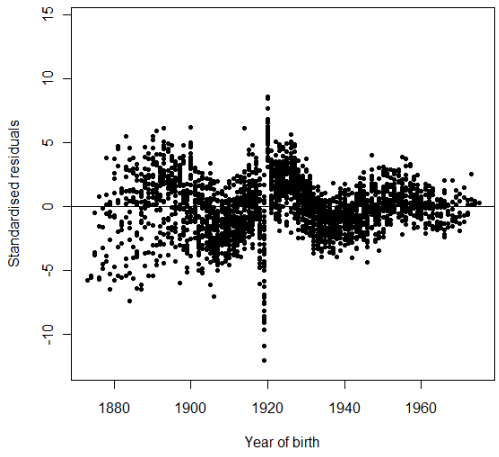
(i)



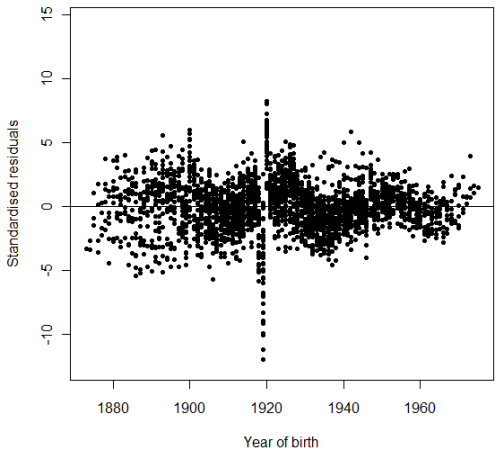
(ii)



(iii)



(iv)



3: Adding a cohort effect to the model

3.1 Introduction

We observed strong evidence of a cohort effect for England and Wales data in Chapter 2. Some of this cohort effect was implicit in the shapes of the parameter graphs under the more complex models, *e.g.* the graphs in Figure 2.9 for male data and Figure 2.13 for female data, and some of it remained in the residuals and is visible in Figure 2.24 for male data and Figure 2.26 for female data. For the CMI data, we also observed evidence of a cohort effect, but it was mainly implicit in the shapes of the parameter graphs, *e.g.* Figure 2.3, rather than remaining in the residuals in Figure 2.22.

It would be desirable to allow for cohort effects via a model that incorporates them directly and places them on an equal footing with period effects. An example of such a model would be:

$$\mu_{xt} = \kappa_t^{(0)} + \gamma_{t-x}^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(3)} + \gamma_{t-x}^{(4)}(x-\bar{x}) + \gamma_{t-x}^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]$$

However, it is found that models of this type fail to produce sensible maximum likelihood estimates of the kappa and gamma parameters. The reasons for this are essentially the same as those in Appendix B, which is referred to below.

Although we shall not fit a model of this type, we should still allow for the cohort effects that have already been observed in certain generations when projecting future mortality rates for those generations. We should also allow in stochastic projections for the risk that future generations will experience either favourable or adverse cohort effects. Accordingly we shall apply a factor depending on year of birth to the force of mortality modelled in Chapter 2. Thus the revised force of mortality at age *x* in year *t* is:

$$\mu_{xt} = \gamma_{t-x}\mu_{xt}^0,$$

where γ_c is a parameter to be estimated for each year of birth *c* and μ_{xt}^0 is the force of mortality modelled in Chapter 2. In view of the findings of Chapter 2, we shall use the GM(1,3) model for the values of μ_{xt}^0.

Ideally, we would use a single iterative procedure to estimate both the gamma and the kappa parameters. However, if this is done, then it is found that towards the extremities of the range of years of birth, the gamma parameters reflect effects that clearly cannot be genuine cohort effects. The gamma parameters instead serve to extend the range of shapes of the mortality curve as a function of age available in each calendar year. Appendix B gives further details of the investigations that were carried out in relation to this. Given this problem, in the estimation of the gamma parameters, we shall assume that the kappa parameters have already been fixed.

3.2 Restricted data sets

Consistently with the approach of Cairns *et al*. (2007), it was considered that there were insufficient data to estimate the gamma parameters reliably for years of birth observed for less than five years in the data. Accordingly, the data relating to lives born before 1861 or after 1971 were removed from the CMI data set, and the data relating to lives born before 1877 or after 1971 were removed from the England and Wales data sets. In what follows, the resulting data sets will be referred to as the 'restricted data sets'.

Before estimating the gamma parameters, it was then necessary to re-estimate the parameters of the GM(1,3) model, *i.e.* the kappa parameters, based on the restricted data

sets. Graphs of the revised parameter values are shown in Figures 3.1, 3.2 and 3.3, which are revised versions of Figures 2.3, 2.9 and 2.13 respectively. Only the parameter values for calendar years before 1951 and after 2001 have changed for CMI data, and only the parameter values for calendar years before 1966 and after 2001 have changed for England and Wales data. This is what we would expect because these are the only years for which any data have been excluded.

Figures 3.4(i), (ii) and (iii) are the graphs corresponding to Figures 2.18(iii), 2.19(iii) and 2.20(iii) respectively for the GM(1,3) model fitted to the restricted data sets, before introducing gamma parameters. The graphs in Figure 3.4 show white triangles in the top left and bottom right corners, representing the years of birth excluded from the data. Otherwise the graphs are very similar to Figures 2.18(iii), 2.19(iii) and 2.20(iii) respectively, and identical for years of birth 1951-2001 inclusive for CMI data and 1966-2001 inclusive for England and Wales data.

Figures 3.5 and 3.6 are the graphs corresponding to Figures 2.21-2.26(iii) when the GM(1,3) model is fitted to the restricted data sets, before introducing gamma parameters. Again there are no significant changes compared with the unrestricted data sets.

3.3 *γ* parameters (cohort parameters)

As discussed in Section 3.1 and in Appendix B, we have selected a simple method of adjusting for a cohort effect, estimating the gamma parameters, which are applied as multiplicative factors to the fitted forces of mortality, without changing our estimates of the kappa parameters. Specifically, γ_c was estimated, for each year of birth c, as:

$$\hat{\gamma}_c = \frac{\sum_x D_{x,c+x}}{\sum_x E_{x,c+x}\mu^0_{x,c+x}},$$

the maximum likelihood estimate of γ_c subject to the constraint that the kappa parameters are unchanged. This is the ratio of the actual number of deaths to the expected number of deaths under the GM(1,3) model for year of birth c, the 'A/E'.

Figures 3.7(i), (ii) and (iii) show the values of $\hat{\gamma}_c$ for each year of birth c, for CMI data, England and Wales male data and England and Wales female data respectively. The following observations can be made from these figures:

- All three figures show a large increase in $\hat{\gamma}_{1920}$ compared with $\hat{\gamma}_{1919}$. This feature was commented on in Section 2.8. It is slightly smaller than for CMI data than for the England and Wales data sets but still not insignificant.

- All three figures show a fall in the gamma parameters which can be identified with the 1925-45 favourable cohort effect. The fall is larger for England and Wales females than for England and Wales males, which ties in with the observation in Section 2.8 that more of the cohort structure was captured implicitly via the kappa parameters for England and Wales males than for England and Wales females. For CMI data, the fall is smaller than for England and Wales data, consistently with the observation in Section 2.8 that most of the cohort structure is captured implicitly via the kappa parameters.

- Figure 3.7(iii) shows a rise in the gamma parameters centred on 1915, which can be identified with the adverse cohort effect for females born around this time that was mentioned in Sections 2.3 and 2.8. There is evidence of this rise continuing into the early 1920s, on the far side of the 1919-20 discontinuity.

- The gamma parameters for CMI data in particular become volatile for the more recent years of birth and fall to exceptionally low values for the last two years of birth

in the data (1970 and 1971). These values are based on particularly small numbers of deaths and so should not be incorporated uncritically into future projections.

- Further similarities can be seen between the graphs for the two male data sets in addition to those relating to the 1919-20 discontinuity and to the 1925-45 favourable cohort effect. For example, both graphs show a rise in the gamma parameters for years of birth in the 1890s, followed by a fall from 1900 to around 1915.

- The most significant differences between the graphs for England and Wales males and females occur at the earliest years of birth, where the gamma parameters for males are relatively high and those for females are relatively low. This may be a consequence of data quality issues in this region of the data sets.

3.4 AIC and BIC

Table 3.1 shows the impact on both the AIC and BIC of introducing gamma parameters, for all three restricted data sets.

Table 3.1 – Impact on the AIC and BIC of introducing gamma parameters

Data set	With or without gamma parameters?	Maximum log-likelihood	Number of parameters	AIC	BIC
CMI	Without	-16,584.48	236	-16,820.48	-17,550.06
CMI	With	-16,185.71	347	-16,532.71	-17,605.43
E&W Male	Without	-18,277.96	176	-18,453.96	-18,970.60
E&W Male	With	-15,016.88	271	-15,287.88	-16,083.39
E&W Female	Without	-18,669.46	176	-18,845.46	-19,362.10
E&W Female	With	-15,824.09	271	-16,095.09	-16,890.60

Note that the number of data cells, *N*, used in the calculation of the BIC values in Table 3.1 has been reduced from 3,599 to 3,579 for CMI data and from 2,640 to 2,620 for the two England and Wales data sets, as a result of the exclusion of years of birth observed for less than five years in the data.

From Table 3.1, we see that the introduction of a cohort effect into the model has improved both the BIC and the AIC very significantly for England and Wales data for both males and females. For CMI data, however, the improvement in the AIC is much smaller, and there is actually a deterioration in the BIC. As discussed in Sections 2.8 and 3.1, it appears that most of the cohort effects are captured implicitly via the kappa parameters for CMI data, but that this is less true for England and Wales data. In addition, the smaller size of the CMI data set means that the remaining cohort effects will be less statistically significant. Nevertheless, the similarities between the graphs in Figure 3.7 for CMI data and for England and Wales male data suggest that we are justified in including the cohort effect in the model for CMI data.

It should be noted that it would be posssible to improve further on the AIC and BIC figures shown in Table 3.1 for the models with gamma parameters by estimating both the kappa and gamma parameters in a single iterative procedure. This was investigated for England and Wales male data under the two methods of carrying out the single iterative procedure described in Appendix B. It was found that under both methods, the AIC and BIC increased by approximately 600. The significant size of the improvements is consistent with the observations in Chapter 2 regarding cohort effects being reflected implicitly in the shapes of the kappa parameter graphs, as these observations suggest that we should be able to improve the fit significantly by making these cohort effects explicit. Unfortunately, however, the shapes of the parameter graphs given by the single iterative procedures, as shown in Figures B.1 and B.2, are unsuitable for forecasting.

3.5 Standardised residuals

Table 3.2 shows how the sample variances of the standardised residuals are affected by the introduction of gamma parameters.

Table 3.2 – Impact on the sample variances of the standardised residuals of introducing gamma parameters

Data set	Sample variance of standardised residuals without gamma parameters	Sample variance of standardised residuals with gamma parameters
CMI	2.0878	1.9106
E&W Male	4.5321	1.9531
E&W Female	5.1339	2.9200

The values in Table 3.2 for the model without gamma parameters are slightly different from the values in Tables 2.6-2.8 as a result of the exclusion of years of birth observed for less than five years in the data.

Table 3.2 shows that for the England and Wales data sets, a very significant part of the variation in the standardised residuals of the GM(1,3) model is explained by cohort effects. Consistently with the conclusions we drew from the AIC and BIC values in Table 3.1, the part of the variation explained by cohort effects for CMI data is smaller. The amount of variation in the standardised residuals that remains unexplained after introducing gamma parameters is greater for the England and Wales female data set than for the male data sets.

3.6 Residual plots

Figure 3.8 shows the graphs corresponding to those in Figure 3.4 after the introduction of gamma parameters. As expected from our previous observations, the changes are more significant for the England and Wales data sets than for the CMI data set. For England and Wales data, the significant diagonal patterns in Figures 3.4(ii) and (iii) have largely been eliminated in Figures 3.8(ii) and (iii), and for CMI data, the more limited diagonal patterns in Figure 3.4(i) have largely been eliminated in Figure 3.8(i). The introduction of gamma parameters does not help to eliminate the systematic understatement of mortality at ages 60-65 and the systematic overstatement at ages 65-70 in CMI data – these features can still be seen in Figure 3.8(i).

Figures 3.9 and 3.10 respectively show the standardised residuals plotted against age and year of birth for the GM(1,3) model after the introduction of gamma parameters. The corresponding figures before the introduction of gamma parameters are Figures 3.5 and 3.6 respectively. Points will have moved significantly in Figures 3.9 and 3.10 compared with Figures 3.5 and 3.6 for years of birth where the estimated gamma parameters are significantly different from 1, particularly if the expected number of deaths is sufficiently large to make the movement credible.

The graphs in Figure 3.9 have the same general shape as those in Figure 3.5. The main difference is that most of the extreme standardised residuals for England and Wales data that can be seen in Figure 3.5 have been eliminated in Figure 3.9. It would appear that cohort effects explain most of the extreme standardised residuals that existed before introducing gamma parameters. In fact, most of these extreme standardised residuals arose from the 1919-20 discontinuity, which is now reflected in the higher fitted value of γ_{1920} than of γ_{1919}.

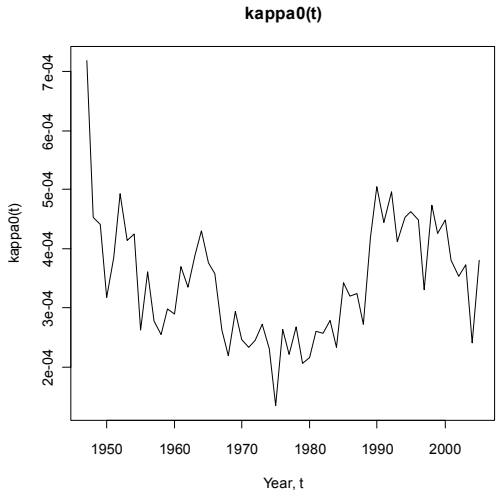
It is clear that Figure 3.10 shows far fewer systematic patterns than Figure 3.6, particularly in England and Wales data where cohort effects not already captured implicitly via the kappa parameters are more significant than for CMI data. In particular, the jumps between years of birth 1919 and 1920 that existed in Figure 3.6 are no longer present in Figure 3.10, because this feature of the data is now reflected in the higher fitted value of γ_{1920} than of γ_{1919}.

3.7 Conclusion

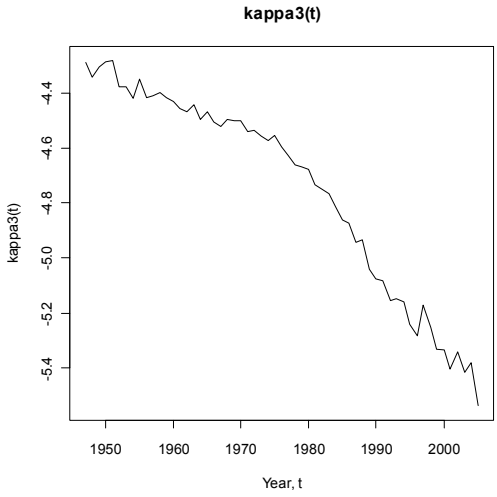
An explicit cohort effect has been added to the GM(1,3) model fitted in Chapter 2. As a result of the need to ensure that the cohort parameters represented genuine cohort effects, a simple approach was selected, in which a multiplicative 'A/E' factor depending on year of birth was applied to the force of mortality under the GM(1,3) model without changing the parameter estimates of the latter. Introducing the cohort effect was found to improve the fit significantly for the England and Wales data sets but not for the CMI data set. However, the view was taken that it was appropriate to include the cohort effect in the model for CMI data as well as for England and Wales data, given the similarities between the fitted cohort parameter values for the different data sets.

Figure 3.1 – Maximum likelihood parameter estimates for the GM(1,3) model fitted to the restricted set of CMI data, before the introduction of gamma parameters **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x}) + \kappa_t^{(5)}((x - \bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$

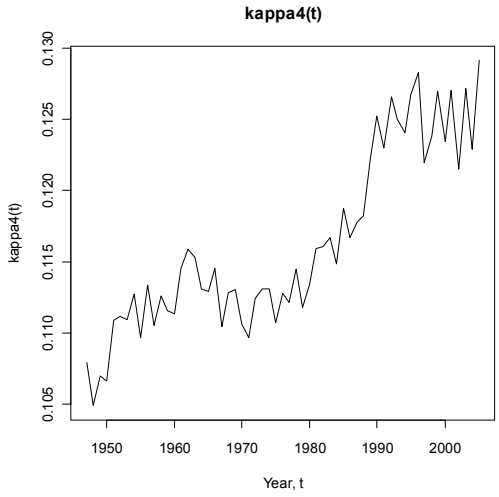
(i)



(ii)



(iii)



(iv)

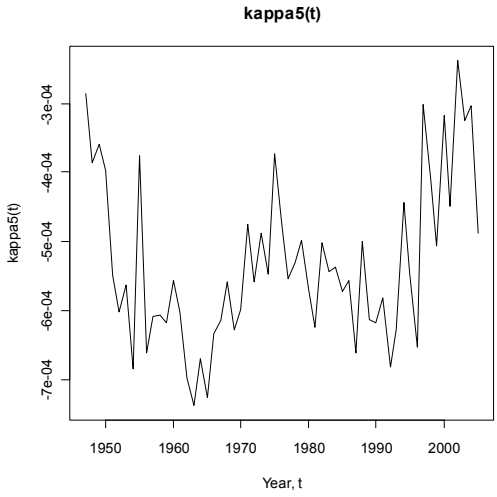
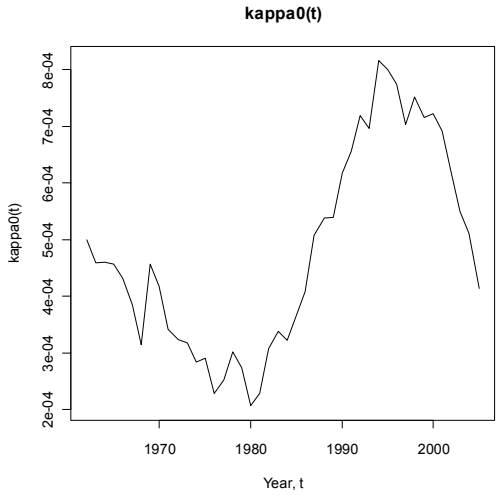
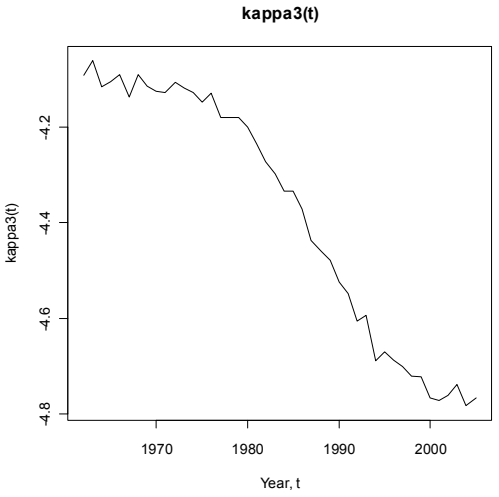


Figure 3.2 – Maximum likelihood parameter estimates for the GM(1,3) model fitted to the restricted set of England and Wales male data, before the introduction of gamma parameters **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x}) + \kappa_t^{(5)}((x - \bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$

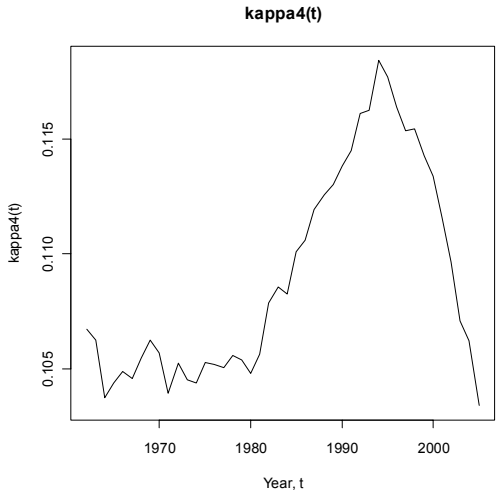
(i)



(ii)



(iii)



(iv)

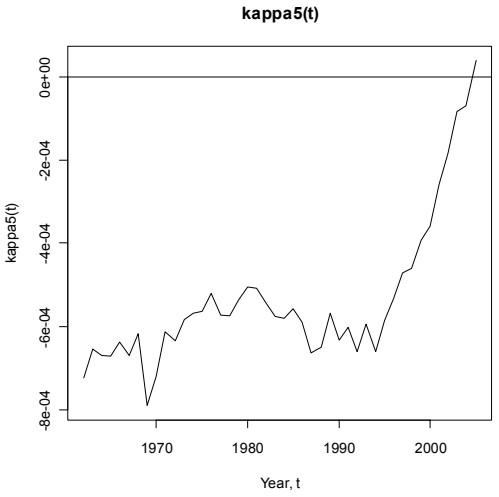
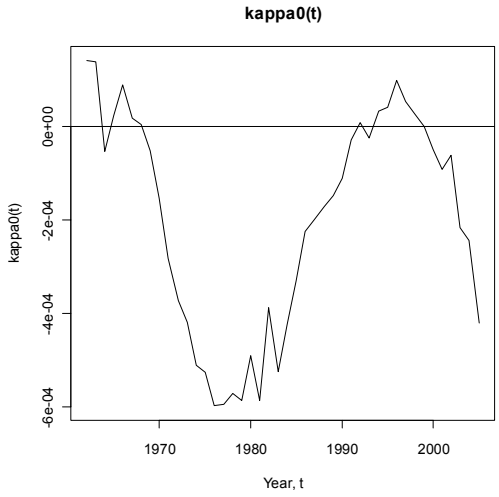
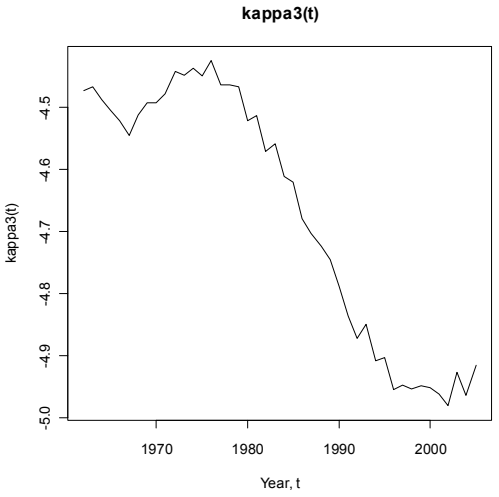


Figure 3.3 – Maximum likelihood parameter estimates for the GM(1,3) model fitted to the restricted set of England and Wales female data, before the introduction of gamma parameters **–** $\mu_{xt} = \kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x - \bar{x}) + \kappa_t^{(5)}((x - \bar{x})^2 - \hat{\sigma}_x^2)]$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$

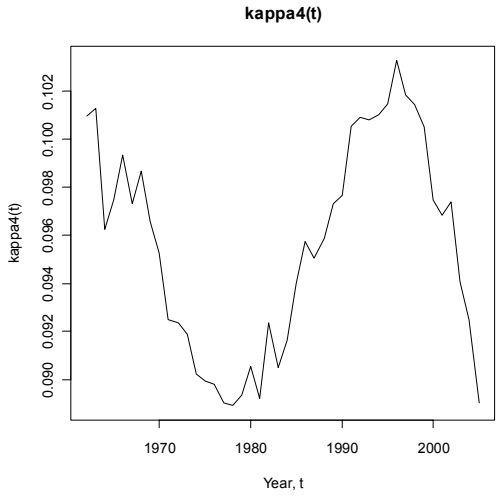
(i)



(ii)



(iii)



(iv)

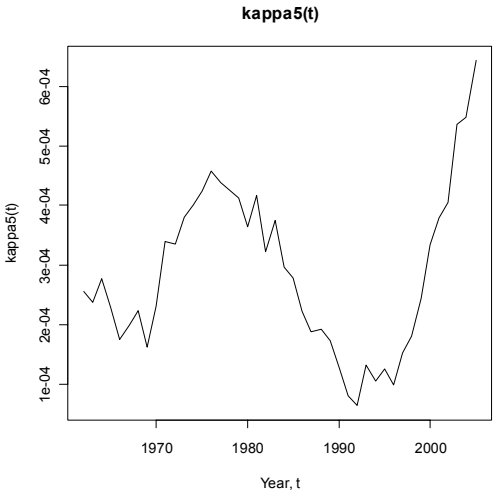
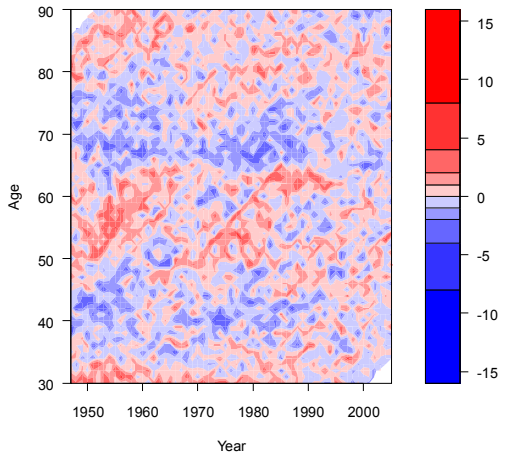
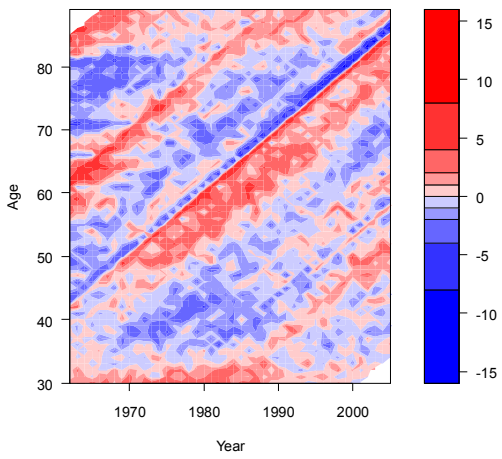


Figure 3.4 – Plots of standardised residuals for the GM(1,3) model fitted to the restricted data sets, before the introduction of gamma parameters **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

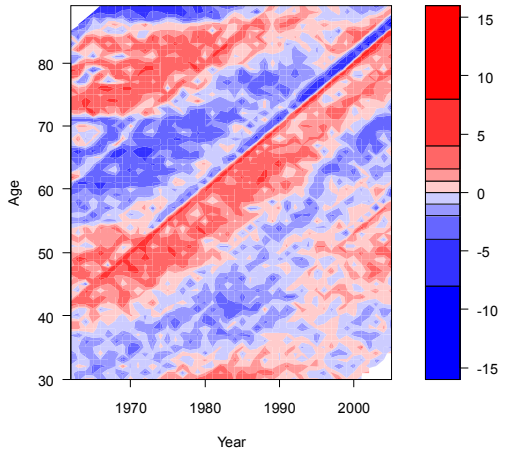
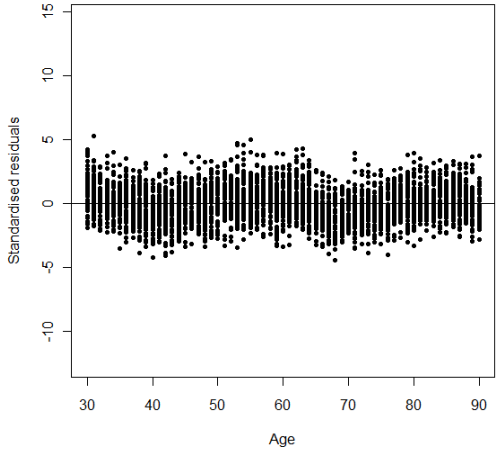
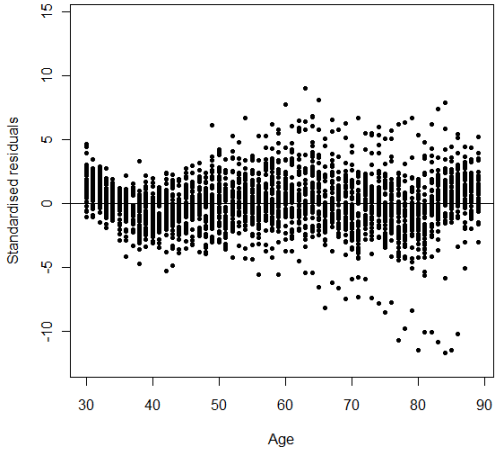


Figure 3.5 – Scatter diagrams of standardised residuals plotted against age for the GM(1,3) model fitted to the restricted data sets, before the introduction of gamma parameters **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

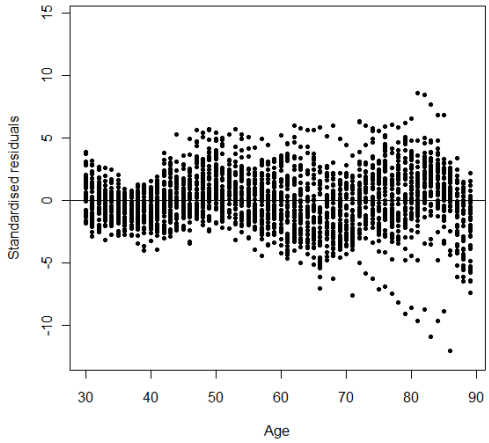
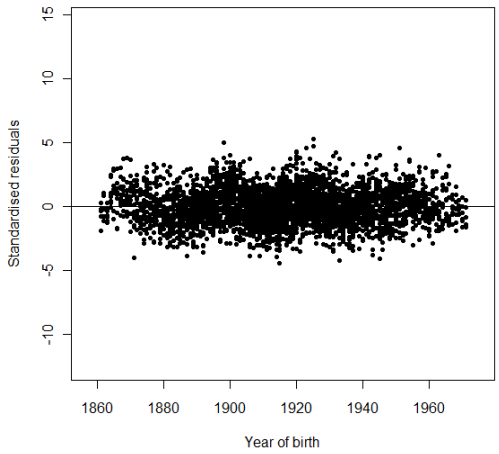
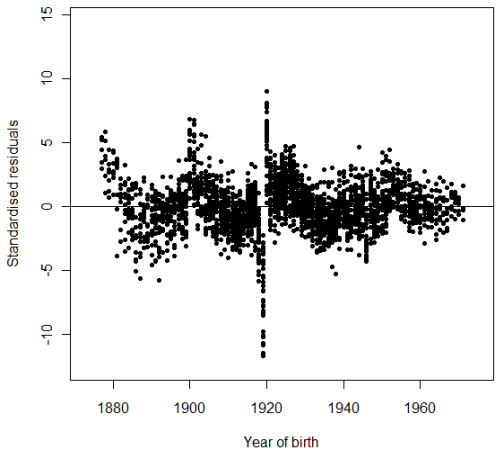


Figure 3.6 – Scatter diagrams of standardised residuals plotted against year of birth for the GM(1,3) model fitted to the restricted data sets, before the introduction of gamma parameters **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

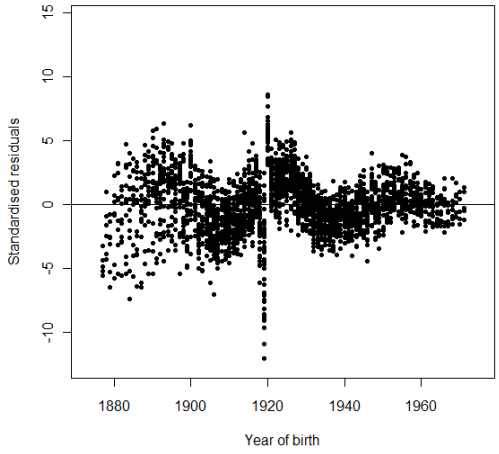
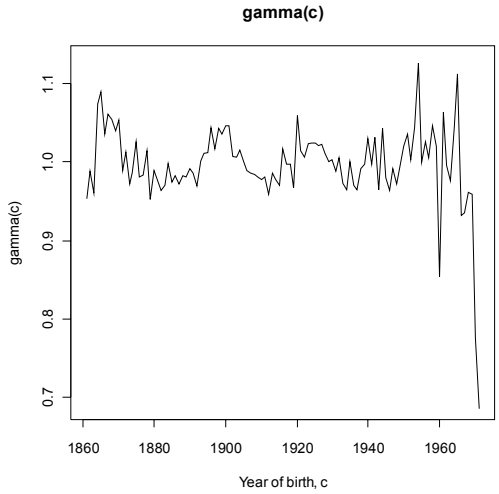
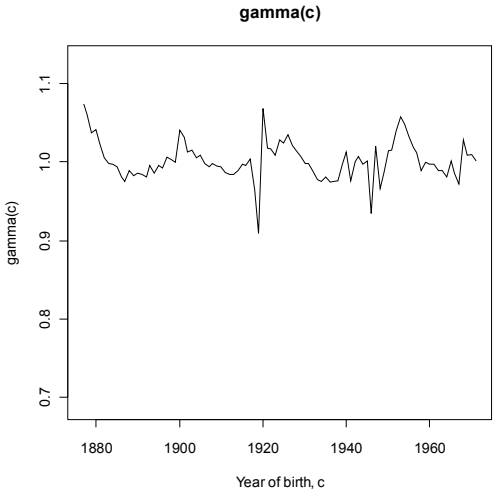


Figure 3.7 – Estimates of the gamma parameters (or equivalently, A/Es for each year of birth) – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

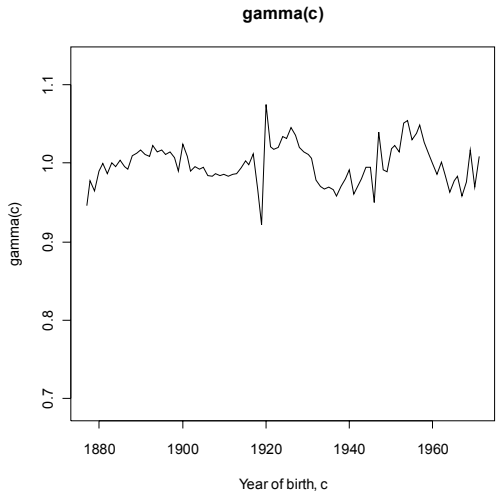
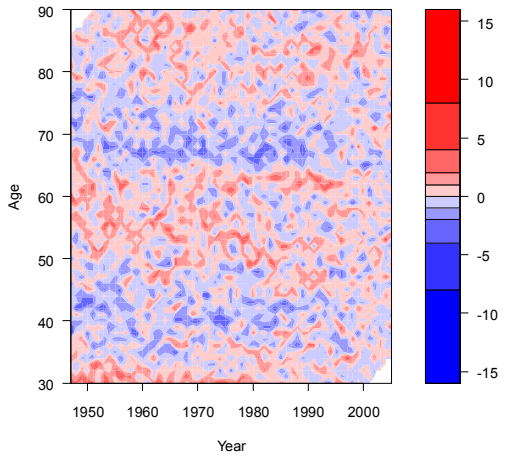
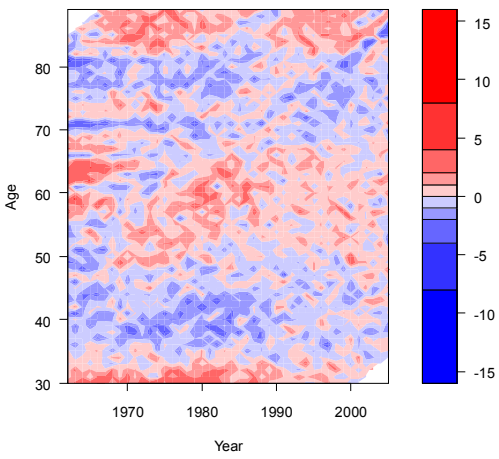


Figure 3.8 – Plots of standardised residuals for the GM(1,3) model extended to incorporate a cohort effect **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

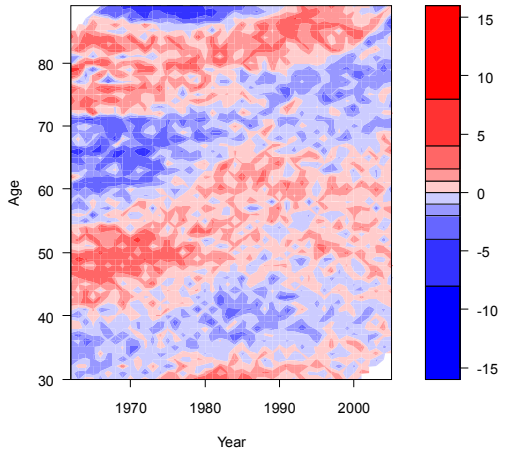
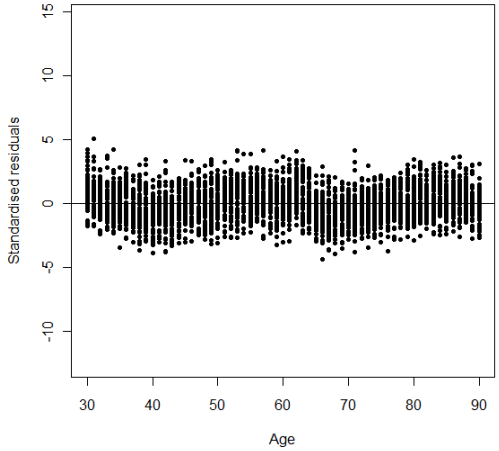
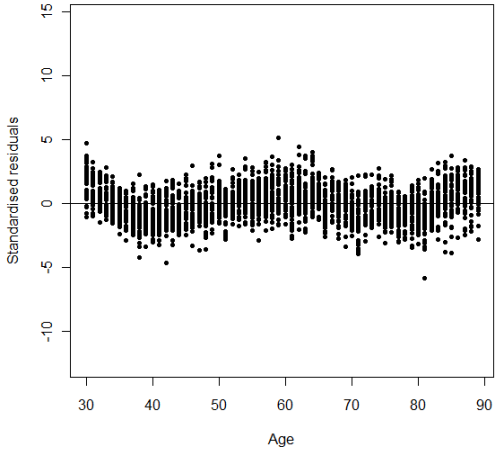


Figure 3.9 – Scatter diagrams of standardised residuals plotted against age for the GM(1,3) model extended to incorporate a cohort effect **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

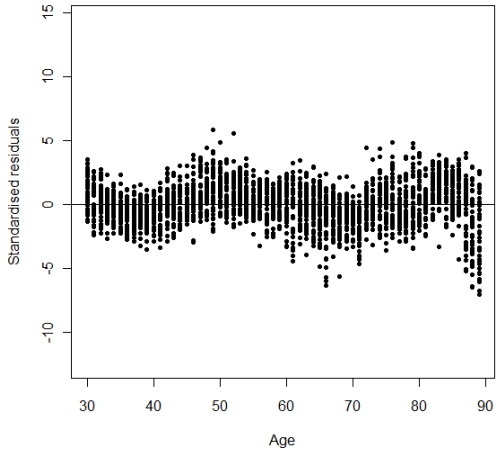
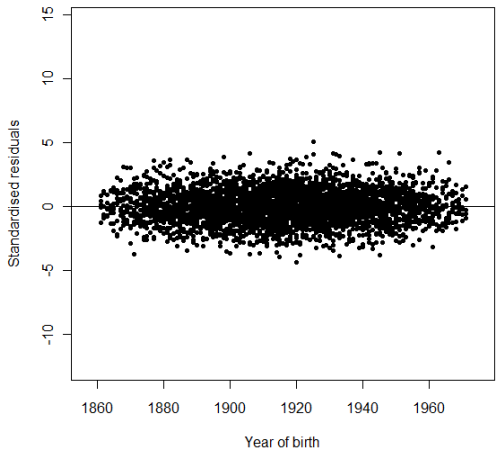
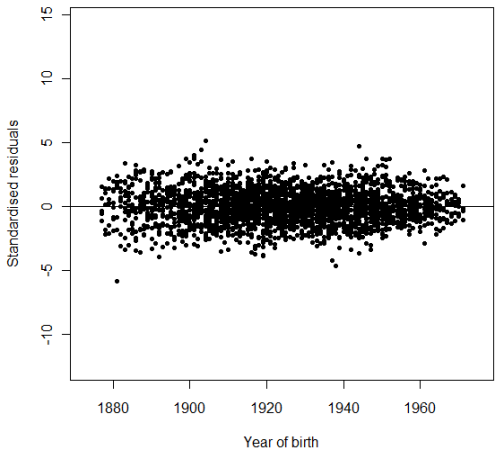


Figure 3.10 – Scatter diagrams of standardised residuals plotted against year of birth for the GM(1,3) model extended to incorporate a cohort effect **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

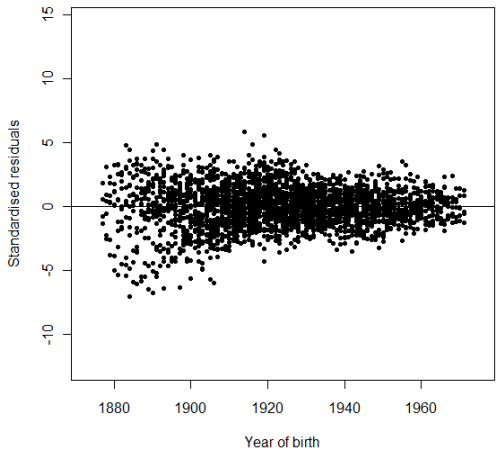
(i)



(ii)



(iii)



4: Fitting time series models to the parameter estimates

4.1 Introduction

In this chapter, time series models are fitted to the estimated parameters of the GM(1,3) model, after introducing gamma parameters, for all three data sets.

We needed to consider cohort effects to explain the shapes of some of the kappa parameter graphs in Chapter 2. None of the time series models we shall consider is capable of incorporating these cohort effects explicitly. Therefore, we shall need to give careful consideration to the impact of cohort effects on each of the kappa parameter graphs before fitting a time series model.

It is to be expected that the parameter estimates of the time series models we fit will be different for the three data sets. If there is clear justification from the data, then it may also be appropriate to fit different *models* to different data sets. However, this should not be done without careful consideration as it might lead to inconsistencies between the future mortality projections for the different data sets, *e.g*. mortality for assured lives might become higher than for the general population in the later years of the projection.

Ideally we would fit a multivariate time series model to all the kappa parameter graphs together. This was investigated but it was found that the cohort effects implicit in the kappa parameter graphs distorted the estimation of the parameters of the multivariate model. Therefore, we shall instead fit univariate time series models to each of the kappa parameter graphs individually, calculate the correlations between the resulting residuals and (in Chapter 5) use these as the correlations between the innovations in future projections. We shall also fit a separate time series model to the gamma parameters.

Appendix C contains the results from the theory of time series analysis that we shall use. However, we note here that a *first-order autoregressive process*, or *AR(1) process*, (X_t) is defined by:

$$X_t = \mu + \alpha(X_{t-1} - \mu) + e_t,$$

where the e_t are independent $N(0,\sigma^2)$ random variables and α, μ and σ^2 are parameters to be estimated, with $|\alpha| < 1$. Under such a process, the values of X tend to revert to the mean level μ, with the reversion being more rapid for small absolute values of α. If $\alpha > 0$, then a value of X above μ tends to be followed by another value of X above μ and vice versa. If $\alpha < 0$, then a value of X above μ tends to be followed by a value of X below μ and vice versa.

A *first-order moving average process*, or *MA(1) process*, (X_t) is defined by:

$$X_t = \mu + e_t + \beta e_{t-1},$$

where the e_t are independent $N(0,\sigma^2)$ random variables and β, μ and σ^2 are parameters to be estimated. If $\beta > 0$, then a value of X_t above μ suggests that X_{t+1} will also be above μ, but subsequent values of X are equally likely to be above or below μ. If $\beta < 0$, then a value of X_t above μ suggests that X_{t+1} will be below μ, but subsequent values of X are equally likely to be above or below μ.

4.2 $\kappa^{(0)}$ parameters

Figures 3.1(i), 3.2(i) and 3.3(i) show no clear signs of a trend over time in the $\kappa^{(0)}$ parameters, with the shapes of Figures 3.2(i) and 3.3(i) being driven mostly by cohort effects. In addition, the $\kappa^{(0)}$ parameters may be thought of as representing mortality from unnatural causes, and it may be reasonable to assume that rates of unnatural death remain broadly constant over time. Thus it is unlikely to be appropriate to fit a model incorporating a trend over time to the $\kappa^{(0)}$ parameters.

Figure 4.1 shows the sample autocorrelation functions, as defined in Section C.1, of the $\kappa^{(0)}$ parameters for the three data sets. All three graphs in Figure 4.1 start with a significant positive autocorrelation at lag 1, which then decays but remains positive for at least the next six lags. As explained in Section C.1, this is a characteristic feature of an AR(1) process. It appears that the $\kappa^{(0)}$ parameters are reverting to some mean level, μ.

Table 4.1 shows the estimates of the parameters α, μ and σ^2 when AR(1) processes are fitted to each of the $\kappa^{(0)}$ series. Figure 4.2 plots the resulting residuals, and Figure 4.3 plots their sample autocorrelation functions.

Table 4.1 – Parameter estimates of the AR(1) processes fitted to the $\kappa^{(0)}$ series

Data set	α	μ	σ^2
CMI	0.7388	3.624×10^{-4}	5.734×10^{-9}
E&W Male	0.9374	4.693×10^{-4}	2.945×10^{-9}
E&W Female	0.9439	-1.738×10^{-4}	6.798×10^{-9}

On inspection of Figure 4.2(i), for CMI data, it seems reasonable to model the residuals as independent $N(0,\sigma^2)$ random variables, where σ^2 has been estimated as 5.734×10^{-9}, and thus the AR(1) model seems reasonable. Figures 4.2(ii) and (iii), for England and Wales data, do show certain patterns in the residuals but they are in line with what we would expect given the known cohort effects.

As per Section C.2, we should regard the autocorrelations shown in Figure 4.3 as statistically significant if their absolute value exceeds $2 / N^{1/2}$, where N is the number of years of data, so that $N = 44$ for the England and Wales data sets and 59 for the CMI data set. Thus autocorrelations exceeding 0.260 in absolute value for CMI data, and exceeding 0.302 in absolute value for England and Wales data, are statistically significant. These critical values are shown as blue dashed lines in Figure 4.3, as they are in all the autocorrelation graphs we shall show. Figure 4.3 shows a few autocorrelations outside the blue dashed lines for each data set but the lags where this occurs are different for each data set and Figure 4.3 does not suggest any key feature of the data which we have failed to model. Accordingly, we shall use the AR(1) model rather than introducing more complex models.

4.3 $\kappa^{(3)}$ parameters

Figures 3.1(ii), 3.2(ii) and 3.3(ii) all show a clear downward trend over time in the $\kappa^{(3)}$ parameters. We should therefore fit a time series model to the differences between the $\kappa^{(3)}$ parameters for successive years, rather than to the $\kappa^{(3)}$ parameters themselves. This will represent the general improvement in mortality over time. Figure 4.4 shows, for each data set, the difference between the $\kappa^{(3)}$ parameter for each year and the parameter for the previous year. Figure 4.5 shows the sample autocorrelation functions of these differences.

Figure 4.5(i), for CMI data, and Figure 4.5(ii), for England and Wales male data, show significant negative autocorrelation at lag 1. The negative autocorrelation does not persist at higher lags. As explained in Section C.1, this is a characteristic feature of an MA(1) process. In Figure 4.5(iii), for England and Wales female data, the negative autocorrelation at lag 1 is not quite statistically significant, but it is reasonable on grounds of consistency with the other data sets to introduce a moving average parameter for England and Wales female data also.

Table 4.2 shows the estimates of the parameters β, μ and σ^2 when MA(1) processes are fitted to each of the series of $\kappa^{(3)}$ differences. Figure 4.6 plots the resulting residuals, and Figure 4.7 plots their sample autocorrelation functions.

Table 4.2 – Parameter estimates of the MA(1) processes fitted to the $\kappa^{(3)}$ differences

Data set	β	μ	σ^2
CMI	-0.4041	-0.02066	0.001697
E&W Male	-0.2257	-0.01609	0.0007734
E&W Female	-0.1067	-0.01050	0.0008256

The model fitted here, *i.e.* an MA(1) model fitted to the differences, is generally referred to as an ARIMA(0,1,1) model in the literature on time series analysis. ARIMA stands for 'autoregressive integrated moving average'. The 0 in (0,1,1) indicates that no autoregressive parameters have been introduced, the first 1 indicates that the time series has been differenced once and the second 1 indicates that one moving average parameter has been introduced.

Figure 4.6 shows some evidence of the residuals being systematically positive until 1980, negative from 1980 to the early 1990s and positive thereafter, particularly for England and Wales data. This is a consequence of the 1925-45 cohort effect, as lives born in 1925-45 have experienced particularly high mortality improvements compared with the previous generation and the mortality of these lives will be driven mainly by the $\kappa^{(3)}$ parameters from 1980 to the early 1990s. However, after considering Figures 3.2(ii) and 3.3(ii), it was concluded that is still reasonable to fit a time series model to the $\kappa^{(3)}$ differences over the full range of years in the data. This is because these figures show a flattening of the $\kappa^{(3)}$ parameters after the early 1990s, as the 1925-45 cohort ages and their mortality is driven more by the $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters. This effectively cancels out the steepening of the fall in the $\kappa^{(3)}$ parameters from 1980 to the early 1990s.

Figure 4.7(i) suggests that the ARIMA(0,1,1) model fits the CMI data well, with no statistically significant autocorrelations of the residuals. For England and Wales data, Figures 4.7(ii) and (particularly) (iii) reveal one potential source of concern, namely the significant positive autocorrelations at lag 2. However, cohort effects appear to have caused the autocorrelations at the shorter lags in these graphs to be systematically positive. If we fitted a model that placed cohort effects on an equal footing with period effects, it seems likely that the autocorrelations at lag 2 would no longer be statistically significant. On balance, therefore, introducing additional complexity into the model for the England and Wales data sets to capture this feature would not appear to be warranted. Thus we shall use the ARIMA(0,1,1) model.

It is of interest to consider what property of the underlying mortality rates the moving average parameter is reflecting. The negative moving average parameter is implying that high mortality in one year has a tendency to be followed by low mortality in the following year, and vice versa. In fact, if mortality is high in a particular year, then it is likely that some external factor driving mortality was adverse in that year, *e.g.* the year may have had a particularly harsh winter. In the following year, not only is it unlikely that this external factor will be equally adverse, but even if it is, mortality is unlikely to be as high because many of the lives who would otherwise have died in the second year died in the first year instead.

4.4 $\kappa^{(4)}$ parameters

We shall consider the most appropriate time series model for the $\kappa^{(4)}$ parameters for England and Wales male data first, then for England and Wales female data and finally for CMI data.

Before 1980, the graph of $\kappa^{(4)}$ parameters for England and Wales male data, Figure 3.2(iii), remains relatively level around 0.105. After 1980, the graph is dominated by the peak in the early 1990s which is a consequence of the 1925-45 cohort effect. By the final years of the data set, the graph has returned to levels around 0.105. Due to the singular nature of the cohort effect, this suggests that it is reasonable to assume a long-term mean of 0.105 for the $\kappa^{(4)}$ parameters.

If the $\kappa^{(4)}$ parameters move away from 0.105 as a result of future cohort effects, then it is likely that they will follow a similar pattern to that of Figure 3.2(iii), moving away over a period of 10-15 years before returning to a level around 0.105 over the following 10-15 years. If we fit an AR(1) process with $\mu = 0.105$ to the $\kappa^{(4)}$ parameters, then we will not be able to replicate this behaviour exactly as the time taken to return to a level around 0.105 will be random. However, for the purpose of quantifying the extent of uncertainty in future mortality rates, such a process is likely to be appropriate.

If we fit an AR(1) process to the $\kappa^{(4)}$ parameters, forcing μ to equal 0.105, then the estimated parameter α is 0.9678. It is to be expected that the estimated value of α is close to 1, as the cohort effect caused the $\kappa^{(4)}$ parameters to be away from 0.105 over an extended period, rather than to move away from 0.105 briefly and then return.

For England and Wales female data, it seems likely that the long-term mean of the $\kappa^{(4)}$ parameters is approximately 0.09. This conclusion has been based mainly on the years since the early 1990s, once the lives born around 1915 who experience an adverse cohort effect have reached an age where they no longer have a significant impact on the $\kappa^{(4)}$ parameters. We see that, since the early 1990s, Figure 3.3(iii) has closely resembled a shifted-down version of Figure 3.2(iii). This suggests that, consistently with the treatment of England and Wales male data, we should use the lower end of the downward trend from the early 1990s to 2005 as our estimate of the long-term mean of the $\kappa^{(4)}$ parameters.

If we fit an AR(1) process to the estimated $\kappa^{(4)}$ parameters, forcing μ to equal 0.09, then the estimated parameter α is 0.9712. Again it is to be expected that the estimated value of α is close to 1.

For CMI data, a first glance at Figure 3.1(iii) suggests that we should be fitting a time series model incorporating an upward trend to the $\kappa^{(4)}$ parameters. However, it does not seem likely that the $\kappa^{(4)}$ parameters will continue to increase indefinitely for assured lives but not for the general population, as this would eventually cause assured lives mortality to exceed population mortality at older ages. Therefore, instead of fitting a time series model incorporating an upward trend, we shall seek to estimate a long-term mean μ of the $\kappa^{(4)}$ parameters and fit an AR(1) process with this value of μ, in a similar way to that for England and Wales data.

Although cohort effects are not as apparent in Figure 3.1(iii) as in Figures 3.2(iii) and 3.3(iii), it is still likely that the apparent trend in Figure 3.1(iii) is in fact mainly a consequence of cohort effects. We indicated in Section 2.8 that the 1925-45 cohort effect is likely to be the explanation for the rise in $\kappa^{(4)}$ parameters from 1980 to the early 1990s, and the overall trend of a rise in the $\kappa^{(4)}$ parameters is in fact heavily concentrated in this period. The more interesting question is why, in CMI data, the $\kappa^{(4)}$ parameters have shown no sign of falling since the early 1990s as in England and Wales data, but the $\kappa^{(3)}$ parameters have instead continued to fall at broadly the same rate as during the period when they were the main driver of the mortality of the 1925-45 cohort.

As a difference has arisen between the behaviour of the parameter graphs for CMI data and for England and Wales data, there must have been a change in the characteristics of the assured lives population relative to the general population. The key change which suggests itself in this case is that UK life insurance companies began to write large volumes of mortgage-related endowment business when tax relief on premiums under qualifying life insurance contracts was introduced in 1979. Although this tax relief was abolished for new business from 1984, insurers continued to write mortgage-related endowment business in significant quantities until the early 1990s. The CMI does not publish assured lives mortality data separately for mortgage-related and non-mortgage-related business, but what we observe regarding the $\kappa^{(3)}$ and $\kappa^{(4)}$ parameters is consistent with mortgage-related business exhibiting lower mortality than non-mortgage-related business. This is because, as the mortgage endowment policyholders reach ages around 60, the $\kappa^{(3)}$ parameters can then be expected to fall. The $\kappa^{(4)}$ parameters will then tend to rise to maintain broadly the same level of mortality at older ages, by which the mortgage endowments have mostly matured. We also observe that the $\kappa^{(0)}$ parameters did not increase as much in the 1980s for CMI data as for England and Wales data, and this is consistent with mortgage endowment policyholders of working age experiencing low mortality.

If the above is the correct explanation of why the $\kappa^{(3)}$ parameters have continued to fall steeply, and the $\kappa^{(4)}$ parameters have not fallen, since the early 1990s, then the phenomenon is likely to be temporary as most of the mortgage endowment business is due to mature within the next 5-10 years. It is therefore likely that the falls in the $\kappa^{(3)}$ parameters will slow down to a rate more in line with historical rates. The $\kappa^{(4)}$ parameters will then tend to decrease, so that

mortality in the early years of retirement continues to improve at a similar rate to that in the past. Thus we conclude that the $\kappa^{(4)}$ parameters are likely to fall back to the levels of the 1960s and 1970s. We shall fit an AR(1) process to the estimated $\kappa^{(4)}$ parameters forcing μ to equal 0.1125, which is approximately equal to the mean value of the $\kappa^{(4)}$ parameters over the 1960s and 1970s.

We cannot estimate the parameter α of this AR(1) process by reference to the whole period 1947-2005, as the estimation procedure will fail to take account of the fact that the $\kappa^{(4)}$ parameters can be expected to fall back to a level around 0.1125 after 2005. Neither is it likely to be appropriate to estimate α by reference only to the period 1947-1980, as this would not consider the possibility of the $\kappa^{(4)}$ parameters being away from 0.1125 for an extended period of time because of cohort effects. Instead we shall use the same value of α as for England and Wales males, 0.9678.

Table 4.3 summarises the parameters of the AR(1) processes we have selected. It also shows the estimates of the error variances σ^2.

Table 4.3 – Parameter estimates of the AR(1) processes fitted to the $\kappa^{(4)}$ series

Data set	α	μ	σ^2
CMI	0.9678	0.1125	7.476×10^{-6}
E&W Male	0.9678	0.105	1.446×10^{-6}
E&W Female	0.9712	0.09	3.102×10^{-6}

Figure 4.8 shows the residuals resulting from fitting these AR(1) processes. The residuals for CMI data show no clear pattern. The shapes of the graphs for England and Wales data are generally in line with what we would expect given the known cohort effects, with the residuals tending to be positive over periods where cohort effects cause the $\kappa^{(4)}$ parameters to rise, and negative over periods where cohort effects cause the $\kappa^{(4)}$ parameters to fall. It seems unlikely that we would be able to eliminate the patterns remaining in these graphs without using a model placing cohort effects on an equal footing with period effects.

Figure 4.9 shows the sample autocorrelation functions of the residuals in Figure 4.8.

The CMI data show a significant negative autocorrelation at lag 1, which does not persist at subsequent lags, possibly suggesting a need to introduce a moving average parameter. The England and Wales data sets do not show similar effects. Introducing a (negative) moving average parameter for the CMI data only would improve the fit somewhat, but it would make the model for CMI data inconsistent with those for England and Wales data. It is also unlikely to have a significant impact on future projections as, rather than implying a different overall trend in mortality over time, it simply means that mortality increasing rapidly with age in one year tends to be followed by mortality increasing more slowly with age in the following year and vice versa. We shall not introduce such a moving average parameter here.

The England and Wales male data show significant positive autocorrelations for the first few lags, but this is more likely to be related to the systematic patterns in the residuals arising from the cohort effect rather than suggesting a need to introduce a further autoregressive parameter.

4.5 $\kappa^{(5)}$ parameters

The graphs of $\kappa^{(5)}$ parameters in Figures 3.1(iv), 3.2(iv) and 3.3(iv) show a rapid increase in the $\kappa^{(5)}$ parameters since the early 1990s, particularly for England and Wales data, as a consequence of the 1925-45 cohort effect. There is strong reason to believe that this increase will reverse out over the next 10-15 years, as lives in the 1925-45 cohort approach the upper age limit of the data sets. It is therefore necessary to fit a model under which the $\kappa^{(5)}$ parameters revert to a long-term mean, whose estimation should not be biased by cohort effects. This mean reversion should occur over a period of 10-15 years rather than a single year, suggesting the use of an AR(1) rather than an MA(1) model.

Inspection of Figures 3.1(iv) and 3.2(iv) suggests -6×10^{-4} as an appropriate long-term mean for the $\kappa^{(5)}$ parameters, based on the period up to the early 1990s. We shall therefore fit AR(1) models to the $\kappa^{(5)}$ parameters for both CMI data and England and Wales male data, forcing the long-term mean μ to equal -6×10^{-4}.

We cannot estimate the autoregressive parameter α directly by reference to the values in Figures 3.1(iv) and 3.2(iv), as this will not allow for the fact that the $\kappa^{(5)}$ parameters can be expected to fall back to a level around -6×10^{-4} after 2005. However, a cohort effect would be expected to cause the $\kappa^{(5)}$ parameters to deviate from their long-term mean over a similar number of years to that for the $\kappa^{(4)}$ parameters. We shall therefore use the same value of α as for the $\kappa^{(4)}$ parameters, $\alpha = 0.9678$.

For England and Wales female data, it seems likely that the long-term mean of the $\kappa^{(5)}$ parameters is approximately 1×10^{-4}, which is approximately equal to the minimum value the $\kappa^{(5)}$ parameters reach in the early 1990s. It is not immediately obvious that the combined impact on the $\kappa^{(5)}$ parameters of the adverse cohort effect centred on 1915 and the favourable cohort effect centred on 1931 will be broadly neutral over this period. However, the fall in $\kappa^{(4)}$ parameters since the early 1990s, which should not be significantly affected by the adverse cohort effect centred on 1915, is approximately 90% of that for the male data, and this suggests that, in the absence of the adverse cohort effect, the rise in $\kappa^{(5)}$ parameters since the early 1990s due to the cohort effect centred on 1931 would also be approximately 90% of that for male data, which is in fact close to the rise we observe.

Again the parameter α cannot be estimated directly from the values in Figure 3.3(iv), so we shall use the same value as for the $\kappa^{(4)}$ parameters, $\alpha = 0.9712$.

Table 4.4 summarises the parameters of the AR(1) processes we have selected. It also shows the estimates of the error variances σ^2.

Table 4.4 – Parameter estimates of the AR(1) processes fitted to the $\kappa^{(5)}$ series

Data set	α	μ	σ^2
CMI	0.9678	-6×10^{-4}	1.291×10^{-8}
E&W Male	0.9678	-6×10^{-4}	3.742×10^{-9}
E&W Female	0.9712	1×10^{-4}	2.852×10^{-9}

Figure 4.10 shows the residuals arising from fitting these AR(1) processes. The residuals for CMI data show no clear pattern. The patterns that can be seen for England and Wales data can largely be explained by the known cohort effects, so it seems unlikely that we would be able to eliminate these patterns without using a model placing cohort effects on an equal footing with period effects.

Figure 4.11 contains graphs of the sample autocorrelation functions of the residuals in Figure 4.10. For CMI data, there is a similar issue to that for the $\kappa^{(4)}$ parameters in that there is a significant negative autocorrelation at lag 1, but the additional complexity of introducing a moving average parameter to accommodate this was again not considered to be justified. For the England and Wales male data, Figure 4.11(ii) does not suggest any justification for introducing additional parameters. Figure 4.11(iii), for England and Wales female data, shows a significant positive autocorrelation at lag 2, but similarly to Figure 4.7(iii), it is doubtful whether this autocorrelation would still be significant if a model were fitted that placed cohort effects on an equal footing with period effects, and so we shall not introduce additional complexity by attempting to model this feature.

4.6 γ parameters (cohort parameters)

The procedure we used to fit the model involved setting all the gamma parameters to 1 initially, estimating the kappa parameters, and then estimating the gamma parameters as A/Es without changing the kappa parameter estimates. This procedure places a limit on the period for which the gamma parameters can be away from 1. For example, if there were any calendar year such that the gamma parameters were less than 1 for all years of birth that

were within the age range of the data in that year, then the kappa parameters for that year could not maximise the log-likelihood of the model without gamma parameters – it must be possible to increase the log-likelihood by increasing both the $\kappa^{(0)}$ and $\kappa^{(3)}$ parameters for that year in such a way that all the fitted forces of mortality for that year increase by a multiplicative constant. Therefore, we should represent the gamma parameters by a time series model with the property of mean reversion to 1.

We commented in Section 3.3 that the estimates of the gamma parameters for the last few years of birth in the restricted set of CMI data are volatile, and are likely to be unreliable because they are based on small numbers of deaths. For example, if the mean number of deaths in a particular year of birth is 1,000, then the variance is also 1,000 because the variance of a Poisson distribution is equal to its mean. Thus the standard deviation is $(1{,}000)^{1/2} = 31.6$, or 3.16% of the mean. This corresponds to a difference of 0.0316 in the estimated gamma parameter value, so random fluctuations potentially distort Figure 3.7 significantly in years of birth where there are fewer than 1,000 deaths. We shall therefore regard estimates of the gamma parameters as unreliable if they are based on fewer than 1,000 deaths.

In the CMI data, there are fewer than 1,000 deaths for each year of birth from 1956 onwards. (There are also fewer than 1,000 deaths for the year of birth 1861 but this should not have material implications for the reliability of future projections.) In the England and Wales data for both males and females, there are over 1,000 deaths for each year of birth, with the one exception of females born in 1971, where the number of deaths is 958 which is only marginally less than 1,000. Accordingly, we shall fit a time series model to the gamma parameters for CMI data based on years of birth 1861-1955 only. We shall take the gamma parameters for subsequent years of birth from projections, ignoring the values calculated from the data. We shall fit time series models to the gamma parameters for England and Wales data based on the full range of years of birth in the restricted data sets, 1877-1971.

We should like to ensure that there is no possibility, however remote, of the gamma parameters becoming negative in future projections, so that the forces of mortality do not become negative in future projections. We shall therefore fit a time series model to the logarithms of the gamma parameters rather than to the gamma parameters themselves. These logarithms are illustrated in Figure 4.12. We then need a time series model with the property of mean reversion to 0 rather than to 1. As we have observed cohort effects that move the gamma parameters away from 1 over a period of several years rather than for just a single year, we shall use an AR(1) model rather than an MA(1) model.

We should consider the treatment of the 1919-20 discontinuity relating to the influenza epidemic. We have chosen to retain it in the data on the grounds that similar epidemics may occur in the future. An alternative approach, which we do not pursue here but which constitutes a possible area for future research, would be to exclude the discontinuity from the data used to fit the main part of the model but to extend the model to incorporate shocks either in a particular year or for lives born in a particular year.

Table 4.5 shows the estimates of the parameters α and σ^2 when AR(1) processes with $\mu = 0$ are fitted to each of the logged γ series. Figure 4.13 plots the resulting residuals, and Figure 4.14 plots their sample autocorrelation functions.

Table 4.5 – Parameter estimates of the AR(1) processes fitted to the logged γ series

Data set	α	μ	σ^2
CMI	0.4182	0	8.008×10^{-4}
E&W Male	0.4378	0	5.115×10^{-4}
E&W Female	0.4923	0	5.118×10^{-4}

We are now fitting a time series model to 95 rather than 59 data items for CMI data, and to 95 rather than 44 data items for England and Wales data. The threshold absolute value above which we should regard autocorrelations in Figure 4.14 as statistically significant has therefore decreased to $2 / (95)^{1/2} = 0.205$, for both CMI data and England and Wales data.

Figure 4.14 shows that this threshold is exceeded at lag 3 only for CMI data and for England and Wales female data, and is not exceeded at any lag for England and Wales male data. Overall, Figures 4.13 and 4.14 do not suggest any need to add further parameters to the AR(1) model.

4.7 Correlations

We expect there to be significant correlations between the residuals of the time series models we have fitted to the different kappa parameter graphs. Cohort effects are an important source of such correlations – for example, the 1925-45 cohort effect caused a decrease in the $\kappa^{(3)}$ parameters together with an increase in the $\kappa^{(4)}$ parameters from 1980 to the early 1990s. It is necessary to allow for these correlations in projecting future parameter values and thus projecting future mortality rates. Tables 4.6, 4.7 and 4.8 show the sample correlation coefficients between the residuals of the different kappa time series models, for CMI data, England and Wales male data and England and Wales female data respectively. As mentioned in Section 4.1, these correlations will be assumed for the innovations in the future projections in Chapter 5.

Table 4.6 – Sample correlation coefficients between the residuals of the time series models fitted to the kappa parameter graphs for CMI data

	$\kappa^{(0)}$	$\kappa^{(3)}$	$\kappa^{(4)}$	$\kappa^{(5)}$
$\kappa^{(0)}$	1			
$\kappa^{(3)}$	-0.5187	1		
$\kappa^{(4)}$	0.6306	-0.5852	1	
$\kappa^{(5)}$	-0.4404	0.5804	-0.6771	1

Table 4.7 – Sample correlation coefficients between the residuals of the time series models fitted to the kappa parameter graphs for England and Wales male data

	$\kappa^{(0)}$	$\kappa^{(3)}$	$\kappa^{(4)}$	$\kappa^{(5)}$
$\kappa^{(0)}$	1			
$\kappa^{(3)}$	-0.6744	1		
$\kappa^{(4)}$	0.6798	-0.3946	1	
$\kappa^{(5)}$	-0.7421	0.5879	-0.6349	1

Table 4.8 – Sample correlation coefficients between the residuals of the time series models fitted to the kappa parameter graphs for England and Wales female data

	$\kappa^{(0)}$	$\kappa^{(3)}$	$\kappa^{(4)}$	$\kappa^{(5)}$
$\kappa^{(0)}$	1			
$\kappa^{(3)}$	-0.7427	1		
$\kappa^{(4)}$	0.8835	-0.5362	1	
$\kappa^{(5)}$	-0.7999	0.7033	-0.7204	1

We shall not consider the possibility of cross-correlation between the kappa residuals and the gamma residuals here. Ideally, all period effects should be captured in the kappa parameters and all cohort effects should be captured in the gamma parameters. In this situation, it becomes clear that it is appropriate to assume no cross-correlation.

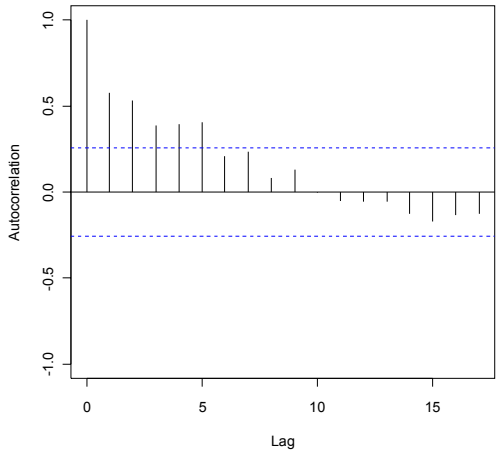
4.8 Conclusion

Univariate time series models have been fitted to the kappa and gamma parameter graphs obtained from fitting the GM(1,3) model, after extension to incorporate a cohort effect, to all three data sets. The models were AR(1) models for the $\kappa^{(0)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters, an ARIMA(0,1,1) model, incorporating the downward trend, for the $\kappa^{(3)}$ parameters and an AR(1) model fitted to the logarithms of the γ parameters. For the $\kappa^{(4)}$, $\kappa^{(5)}$, and γ parameters, the form and parameters of the time series models selected depended fundamentally on a subjective consideration of how cohort effects influenced these parameter graphs, and it is likely that

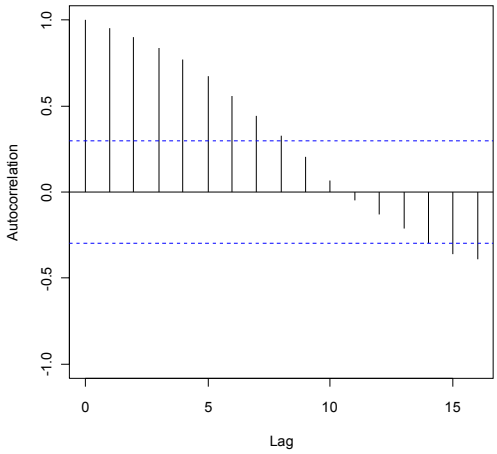
very different models and/or parameter estimates would have been obtained by naïve application of standard fitting techniques. The possibility of fitting a multivariate time series model to all the kappa parameter graphs together was considered, but was not pursued because, in the multivariate case, it was not feasible to give subjective consideration to the impact of cohort effects on the parameter values and naïve application of standard fitting techniques was found not to produce reasonable results. Instead the correlations between the residuals of the time series models for the different kappa parameters were calculated, and these will be used as the correlations between the future innovations in Chapter 5.

Figure 4.1 – Sample autocorrelation functions of the $\kappa^{(0)}$ parameters – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

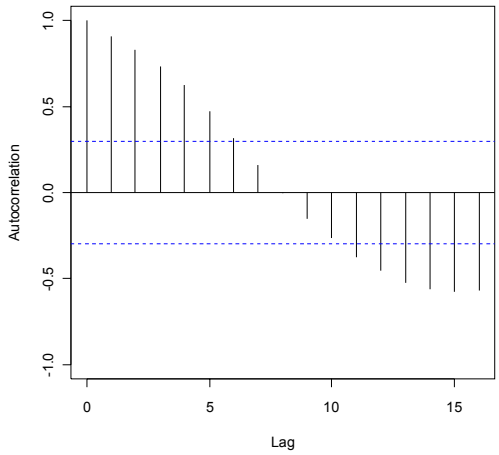
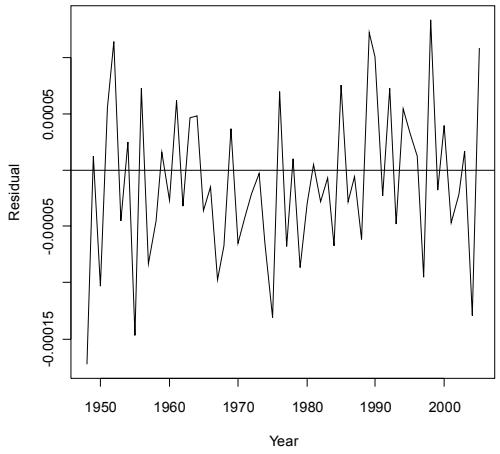
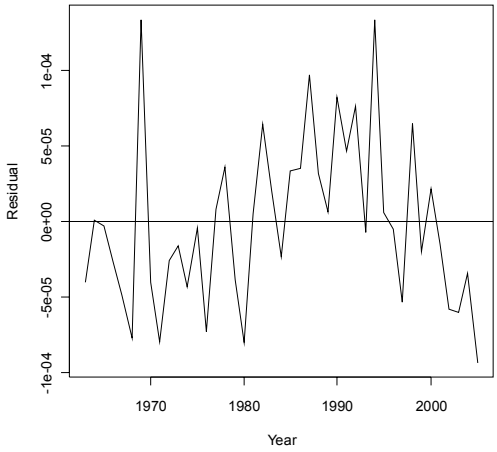


Figure 4.2 – Residuals of the AR(1) processes fitted to the $\kappa^{(0)}$ series – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

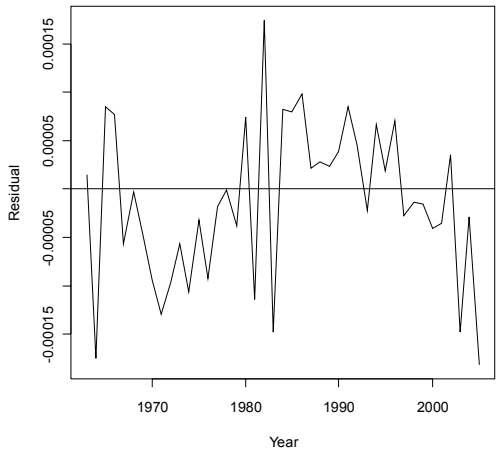
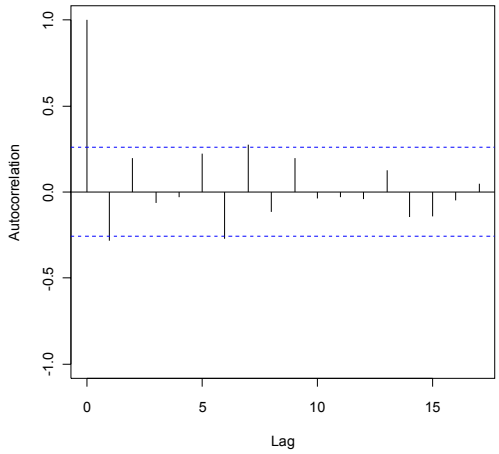
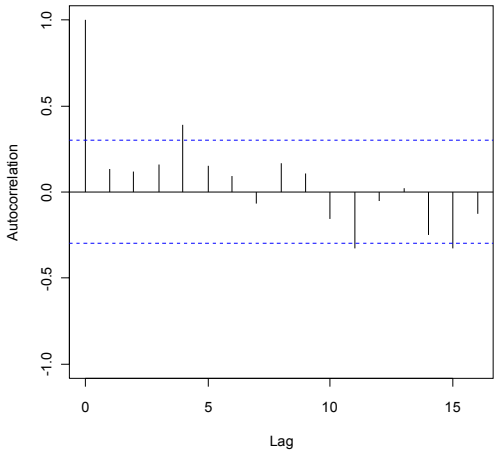


Figure 4.3 – Sample autocorrelation functions of the residuals in Figure 4.2 – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

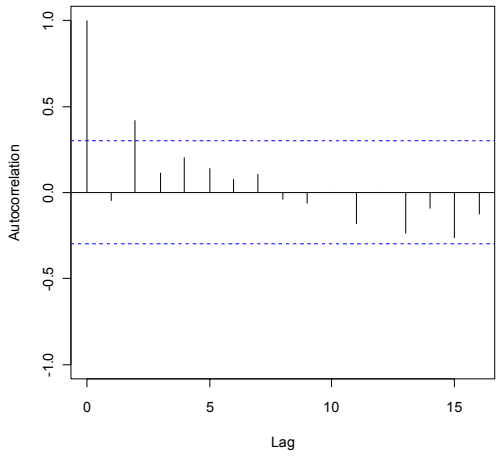
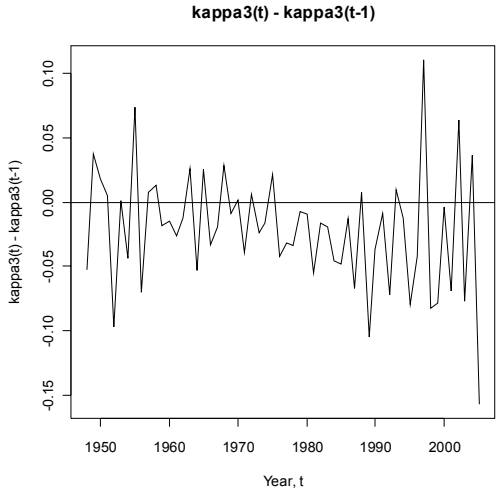
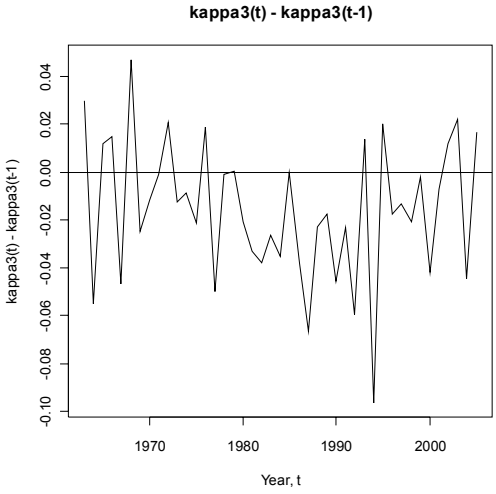


Figure 4.4 – Differences between the $\kappa^{(3)}$ parameters for successive years – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

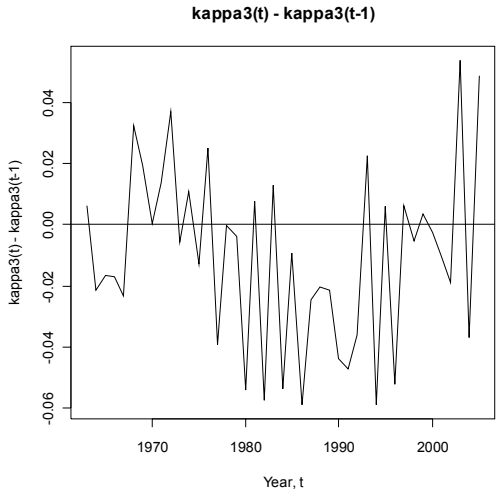
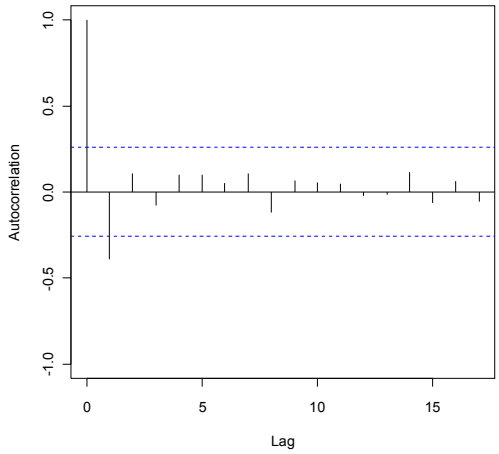
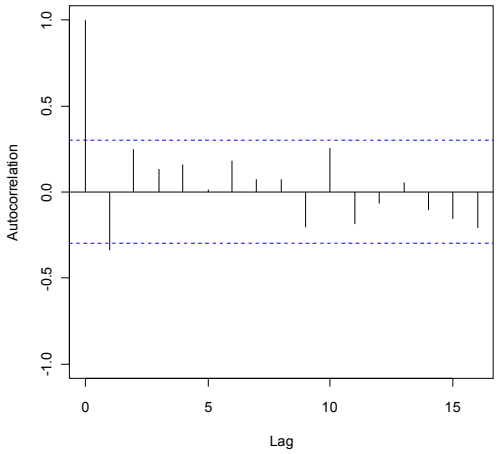


Figure 4.5 – Sample autocorrelation functions of the differences in Figure 4.4 – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

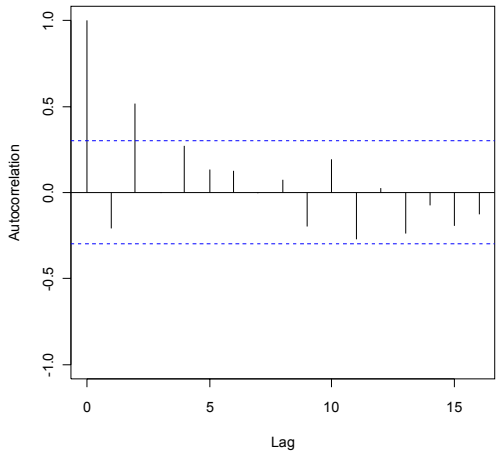
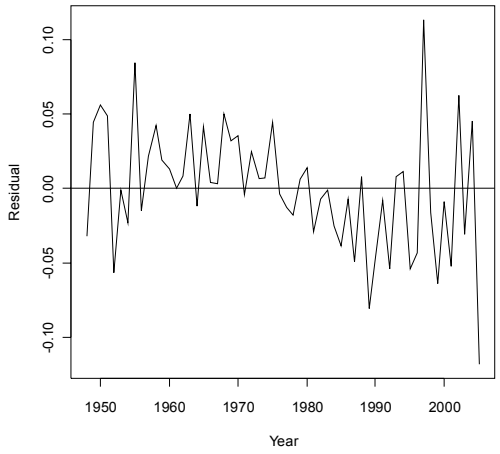
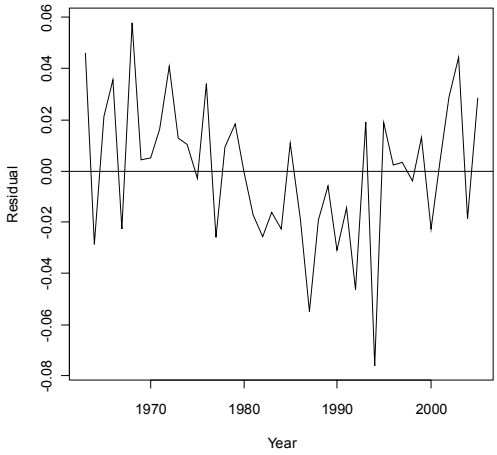


Figure 4.6 – Residuals of the MA(1) processes fitted to the differences in Figure 4.4 – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

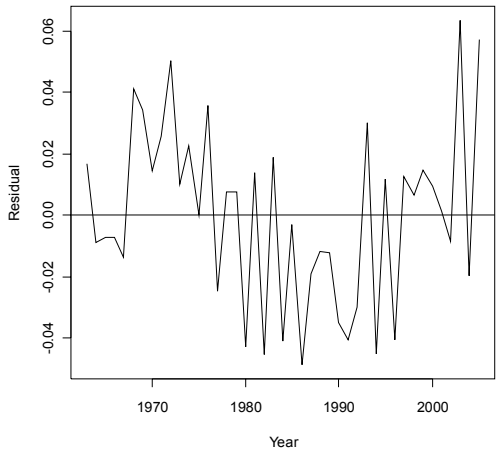
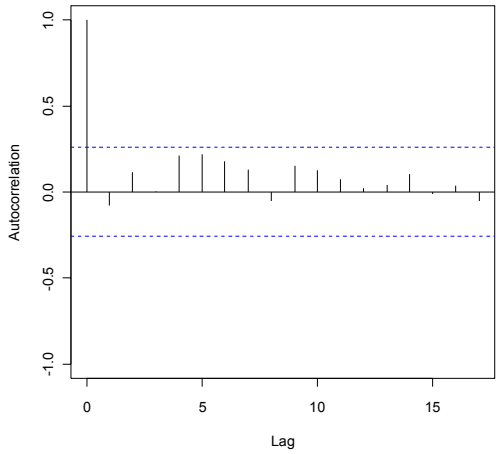
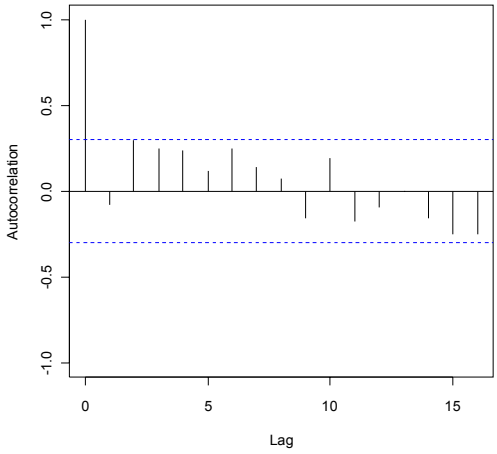


Figure 4.7 – Sample autocorrelation functions of the residuals in Figure 4.6 **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

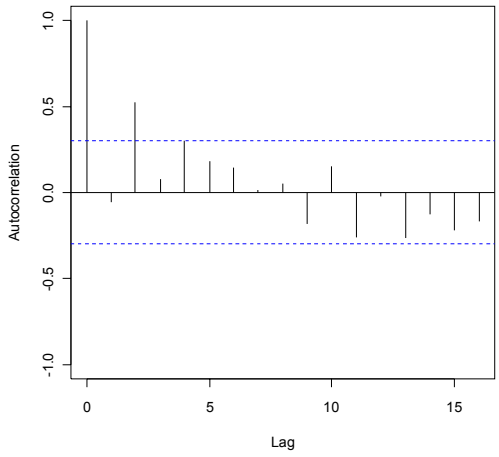
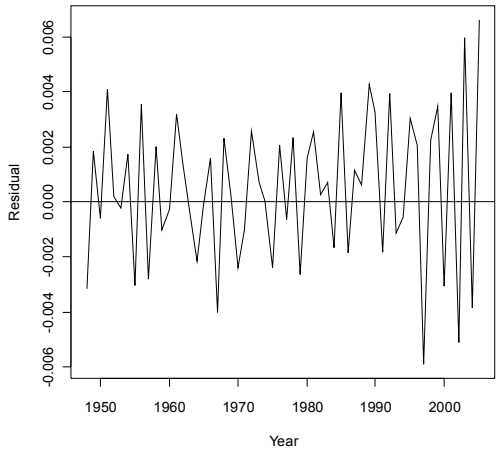
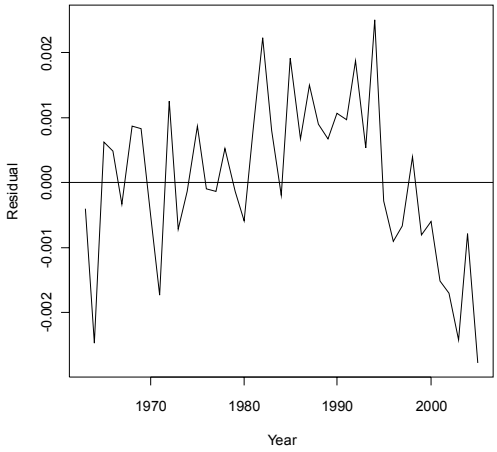


Figure 4.8 – Residuals of the AR(1) processes fitted to the $\kappa^{(4)}$ series – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

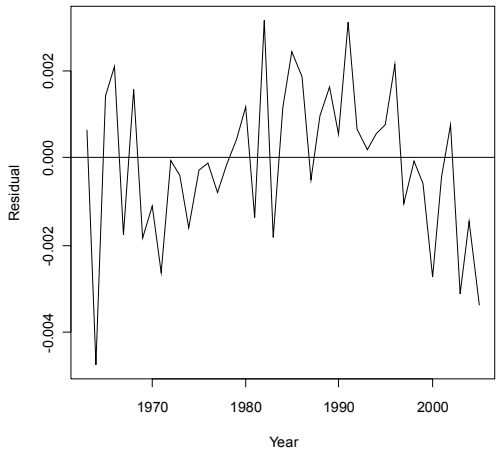
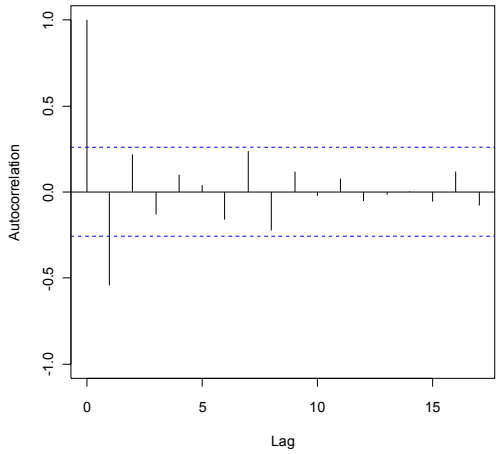
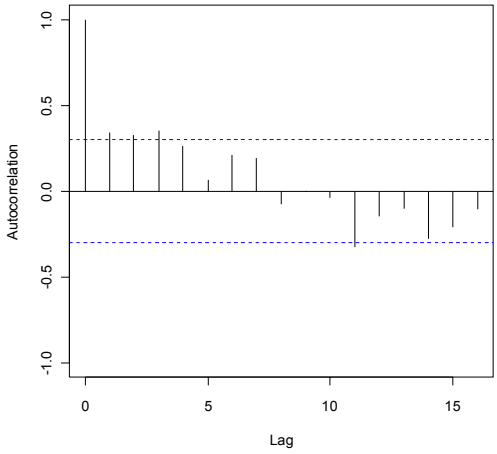


Figure 4.9 – Sample autocorrelation functions of the residuals in Figure 4.8 – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

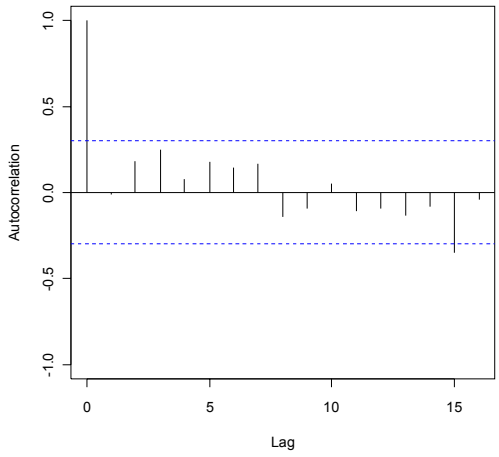
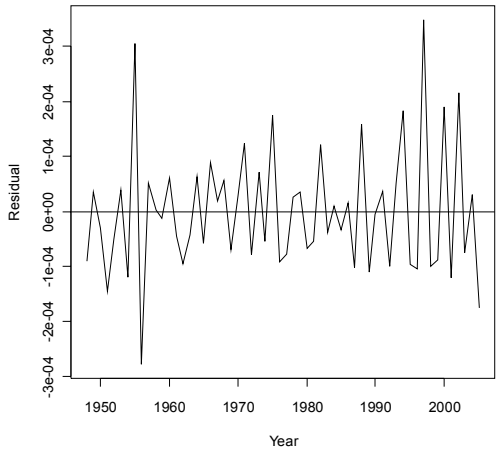
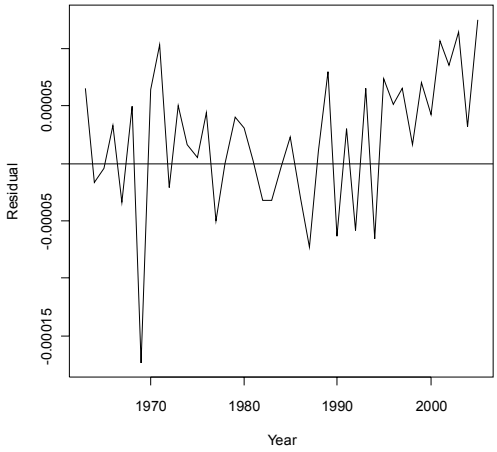


Figure 4.10 – Residuals of the AR(1) processes fitted to the $\kappa^{(5)}$ series – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

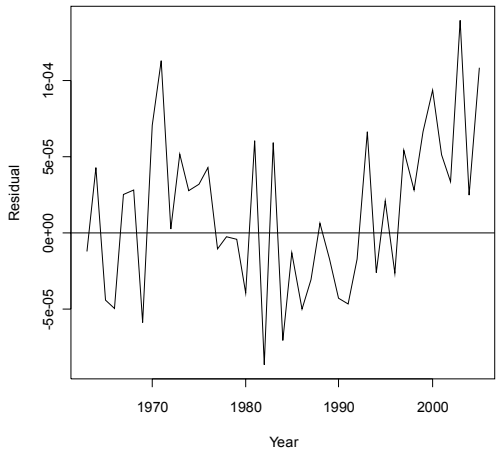
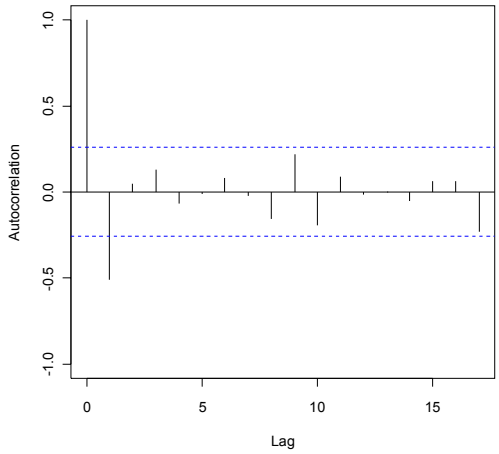
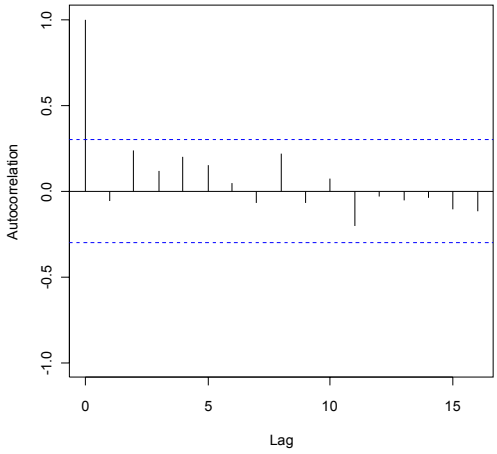


Figure 4.11 – Sample autocorrelation functions of the residuals in Figure 4.10 – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

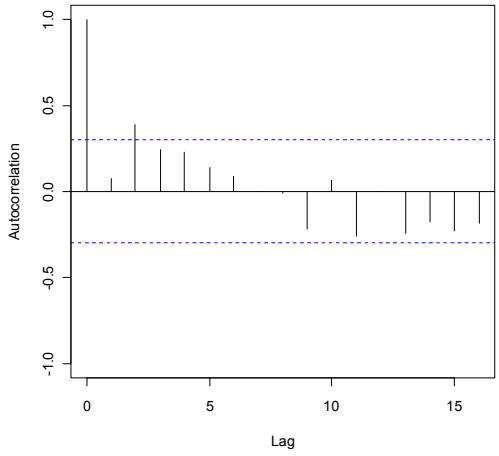
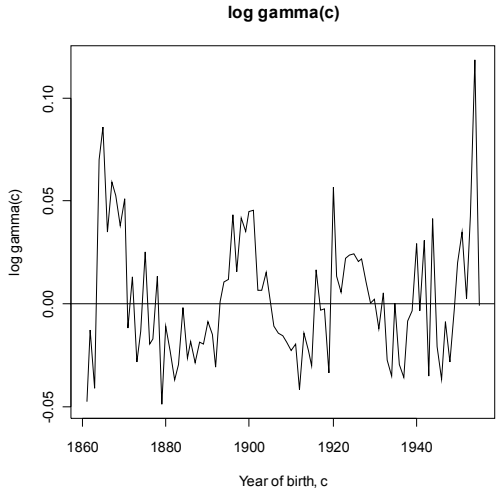
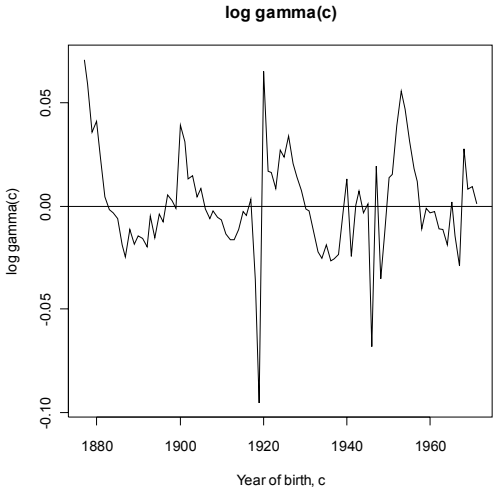


Figure 4.12 – Logarithms of the γ parameters – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

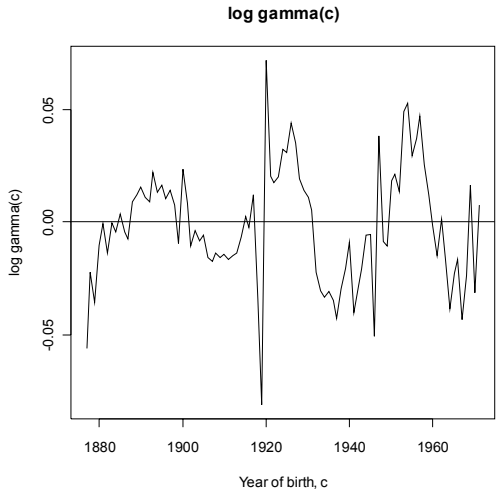
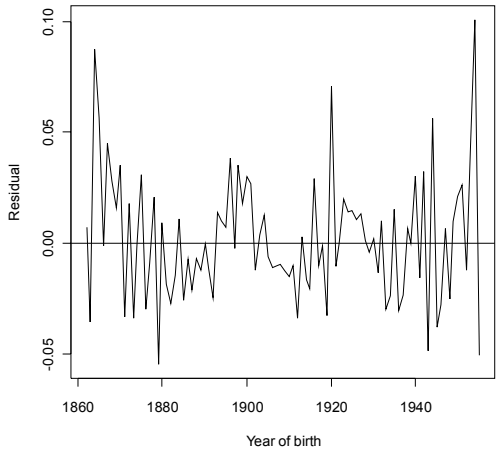
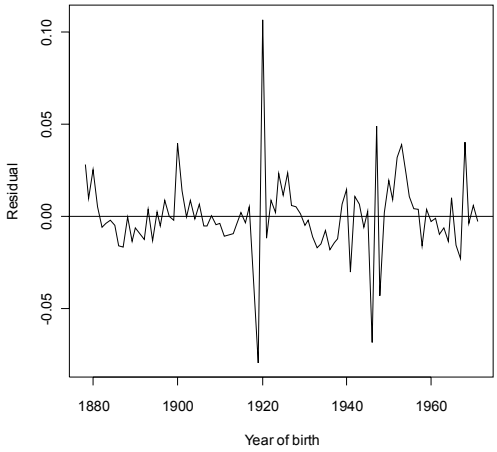


Figure 4.13 – Residuals of the AR(1) processes fitted to the logged γ series – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

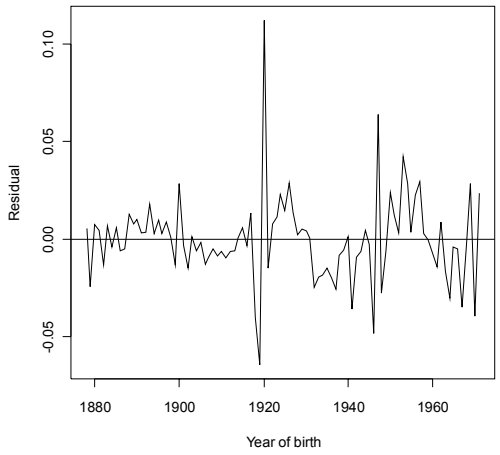
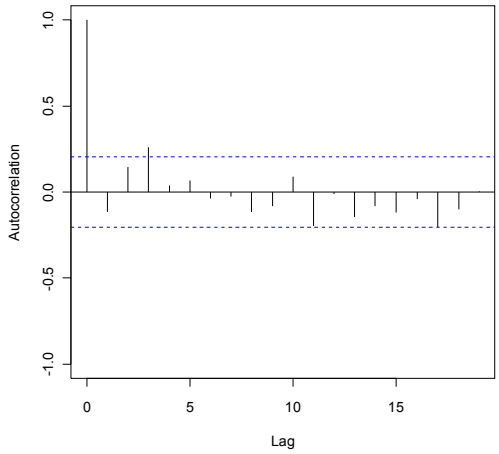
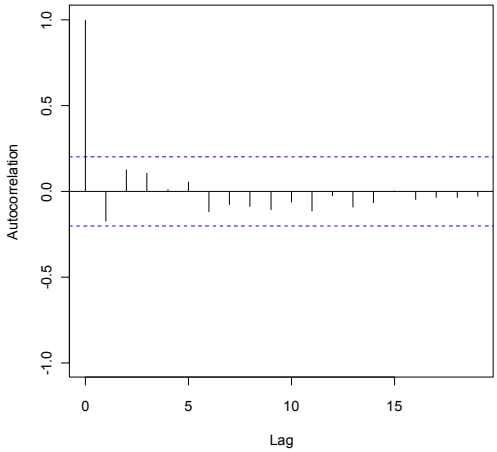


Figure 4.14 – Sample autocorrelation functions of the residuals in Figure 4.13 – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

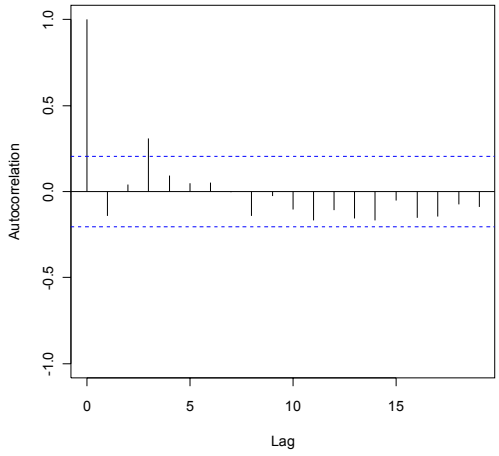
(i)



(ii)



(iii)



5: Forecasting and life insurance applications

5.1 Introduction

For each of the data sets, under the time series models fitted in Chapter 4, 500 sets of sample paths of the future parameter values were generated. The same 500 sets of future normal variates were used for all three data sets. The correlations assumed between the different kappa innovations were those in Tables 4.6-4.8.

Section 5.2 shows prediction intervals to illustrate the extent of uncertainty in the future parameter values. We do not base these prediction intervals on the simulation output, as closed formulae exist for them.

The relationship between the uncertainty in the future parameter values and the uncertainty in projected forces of mortality is relatively complex because of the significant correlations between the different kappa parameter series. We therefore show separate prediction intervals for projected forces of mortality in Section 5.3. These prediction intervals are based on the simulation output. Section 5.3 also compares these projected forces of mortality with a number of previously published projections.

In order to use these forces of mortality to calculate annuity values, it is necessary to extrapolate the mortality curves to ages above the highest age in the data, *i.e.* above 90 for CMI data and 89 for England and Wales data. We identify two methods that might be considered appropriate for extrapolating the mortality curve for 2005 (the latest year in the data). These methods are, however, unlikely to be appropriate for extrapolating projected mortality curves in future years, and so a separate method is required for determining future improvements at ages above the highest age in the data. Two possible methods are considered for this as well.

Section 5.5 uses the projected forces of mortality to calculate immediate and deferred annuity values for lives at different ages in 2005. These annuity values are compared with values from previously published projections.

Section 5.6 applies the projections to calculate risk-based capital requirements for the same immediate and deferred annuities as in Section 5.5, under the Individual Capital Assessment (ICA) regime of the UK insurance regulator, the Financial Services Authority (FSA). The capital requirements calculated are found to be significantly lower than those given by a 'rule of thumb' widely used within the UK life insurance industry, and the implications of this are discussed. Section 5.7 concludes.

5.2 Prediction intervals for the future parameter values

Figures 5.1-5.5 show 95% prediction intervals for the projected parameter values, based on the 2.5th and 97.5th percentiles of the distributions of those values. Both the kappa (period) and gamma (cohort) parameters are shown. This gives the range within which a particular parameter for a particular future year lies under 95% of scenarios. As stated in Section 5.1, these prediction intervals were calculated analytically rather than being based on the simulation output. Section C.3 details the formulae used.

The following observations can be made from Figures 5.1-5.5:

- In Figures 5.1, 5.4 and 5.5, the prediction intervals broadly correspond to the range of values that the $\kappa^{(0)}$, $\kappa^{(5)}$ and γ parameters (respectively) have exhibited in the past. This is because we have fitted a time series model not incorporating a trend to these parameters.

- In Figure 5.3(i), for CMI data, the prediction interval broadly corresponds to the range of values the $\kappa^{(4)}$ parameter has taken in the past. In Figures 5.3(ii), for England and Wales male data, and 5.3(iii), for England and Wales female data, the prediction intervals are biased towards the lower end of the range of past values. For both the

England and Wales data sets, we argued in Section 4.4 that future values of this parameter were likely to be around the lower end of the range of past values, and fitted an AR(1) model with a value of μ reflecting this.

- In Figure 5.2, relating to the $\kappa^{(3)}$ parameters, for which we have fitted a time series model incorporating a downward trend, the prediction intervals show a continuation of this trend. Although the long-term average annual rate of this trend is fixed, being equal to the parameter μ of the MA(1) process fitted to the $\kappa^{(3)}$ parameters, there is significant uncertainty in the actual rate of the trend realised over the projection period.

- The prediction intervals are, in most cases, widest for CMI data and narrowest for England and Wales male data, consistently with the relative values of the estimated innovation variances σ^2. It is likely that the relatively small volume of data in the CMI data set has given rise to volatility in the parameter estimates for successive calendar years and years of birth, resulting in an increase in the estimated innovation variances. For England and Wales female data, the existence of two significant cohort effects, one favourable and one adverse, which are not being modelled explicitly leads to more variation in the parameter values being reflected in the innovation terms than for England and Wales male data.

- For the parameters other than the $\kappa^{(3)}$ parameters, the future width of the prediction intervals converges relatively quickly over time to a limiting value. For the $\kappa^{(3)}$ parameters, however, the prediction intervals continue to widen over time, and their width would in fact increase without limit if the projections were extended indefinitely. In other words, under the model we have fitted, the uncertainty about the overall level of mortality increases the further into the future projections are made, but the uncertainty about the shape of the mortality curve far into the future is not very much greater than in the near future.

It is important to note that the boundaries of the prediction intervals shown in Figures 5.1-5.5 are not sample paths. A typical sample path will have high parameter values in some future years and low parameter values in others, and in addition, as a result of the correlations between the different kappa innovations, a typical sample path will have high values of some kappa parameters and low values of others. Consequently, much less than 5% of sample paths will be as extreme, in terms of financial impact, as the boundaries of the prediction intervals.

5.3 Prediction intervals for projected forces of mortality

Figure 5.6 extends the contour plots of the logarithm of the force of mortality in Figure 1.2 to future years, using a deterministic projection with all the future innovation terms set to zero. The fact that the bands of colour in the graphs continue along a broadly linear path indicates that we are forecasting a broadly constant future percentage improvement rate in mortality.

Figures 5.7-5.9 show 95% prediction intervals for future forces of mortality at specific ages, for all three data sets, based on the 2.5^{th} and 97.5^{th} percentiles. Unlike the prediction intervals for the future parameter values calculated in Section 5.2, these prediction intervals are based on the simulation output.

Superimposed on Figures 5.7-5.9 are the future forces of mortality given by a number of previously published projections. These projections were all obtained from version 1.0 of the 'library' of mortality projections issued by the CMI in 2007. They were applied to the 2005 mortality curves fitted in this thesis, before the introduction of gamma parameters, and the gamma parameter for the appropriate year of birth was then applied to the resulting force of mortality, taking the gamma parameter from a deterministic projection of the time series model fitted in this thesis with all the future innovation terms set to zero if the year of birth was after 1955 for CMI data or 1971 for England and Wales data. Further details of the previously published projections are given below.

For CMI data, two of the three previously published projections we consider are the 'Medium Cohort' and 'Long Cohort' projections first proposed in CMI (2002). These projections were issued by the CMI after the favourable cohort effect centred on 1926 had first been observed in CMI data. This cohort effect was not allowed for in the projection originally published with the '92' series of mortality tables in CMI (1999). The Medium Cohort and Long Cohort projections assume higher mortality improvement rates compared with the previous generation than the projection in CMI (1999) for lives born between 1910 and 1942 inclusive. Between 1992 (the base year of the '92' series mortality tables) and 2000, the higher improvement rates apply to the whole of the range of years of birth from 1910 to 1942, but after 2000, the range of years of birth where they apply reduces linearly, reaching zero in 2020 for the Medium Cohort projection and in 2040 for the Long Cohort projection. The range continues to be centred on 1926.

Baxter (2007) argues that the future increases in period life expectancy implied by the CMI projections are too slow compared with what has been observed in the England and Wales population, a separate issue from the cohort effect. Baxter proposes applying an underpin to the improvement rate across all ages and all future years. Thus the third previously published projection we consider for CMI data is the Medium Cohort projection with an underpin of 1% *p.a.* applied to the improvement rate.

The previously published projections we consider for England and Wales data, for both males and females, are those underlying the population projections produced by the Government Actuary's Department (GAD) on the basis of data to 2004 (GAD, 2006). Responsibility for producing the official UK population projections has since been transferred to the Office for National Statistics (ONS). A 'Principal' projection, a 'High Life Expectancy' (HLE) variant and a 'Low Life Expectancy' (LLE) variant were produced. The male projections relate to England, Wales and Northern Ireland, while the female projections relate to the whole of the UK.

The key points to be noted from Figures 5.7-5.9 are as follows:

- For both the male data sets, the model of this thesis predicts higher future mortality improvements, in percentage terms, at older ages than at younger ages. For England and Wales female data, the long-term mortality improvements predicted are broadly independent of age, but faster at the very youngest ages as a result of the negative $\kappa^{(0)}$ parameters.

- For CMI data, the improvements given by the previously published projections are the same at age 30 as at age 50, as the previously published projections as implemented in the 'library' are the same below age 60 as at age 60. Nevertheless, it is clear that the previously published projections give higher mortality improvements than the model of this thesis in the early years of the projection at younger ages, but that this is no longer the case in the later years of the projection, except in the case of age 30 where a 1% underpin is applied to the improvement rate from the previously published projection.

- For lives aged 70 in CMI data, the three previously published projections are broadly in line with the central projection from the model of this thesis in the early years of the projections. This also applies to the Long Cohort projection, although not to the Medium Cohort or Medium Cohort subject to a 1% underpin, for lives aged 90. However, the Medium Cohort and Long Cohort projections only exhibit higher improvements than the projection in CMI (1999) for one particular generation, so it is to be expected that for subsequent generations, these projections will show lower improvements than the model of this thesis. This can be seen in Figure 5.7(iii) and (iv). Imposing a 1% underpin on the improvement rate is not sufficient to close this gap, because the model of this thesis predicts an improvement rate greater than 1% *p.a.*

- The curves representing both the model of this thesis and the previously published projections become smoother when the year of birth being represented passes 1955

for CMI data or 1971 for England and Wales data. This is because the applicable gamma parameters then begin to come from a projection rather than from the data.

- For England and Wales data, all the curves representing the GAD projections eventually become straight lines, with the lines being horizontal for the LLE projections, downward-sloping for the HLE projections and downward-sloping with half the HLE slope for the Principal projections. This is because the GAD projections have the property that after 2029, mortality improves at a constant rate across all ages and all calendar years, the constant rate being 0% *p.a.* for the LLE projections, 1% *p.a.* for the Principal projections and 2% *p.a.* for the HLE projections. All the curves representing the central projections from the model of this thesis, except those at age 30, eventually become close to straight lines, which are reasonably close to being parallel to the green lines, signifying a long-term improvement rate of approximately 1% *p.a.* For males aged 70 and 90, the long-term improvement rate is slightly faster than this.

- The 2.5th and 97.5th percentile curves from the model of this thesis are less smooth than the median curves. This can be attributed to sampling error, which typically has more effect on the estimation of the more extreme percentiles of a distribution for a given sample size.

- The 2.5th percentile curve for England and Wales females aged 30 falls without limit in the late 2030s. This is because, by this point, the force of mortality has become negative in more than 2.5% of scenarios. The structure of the model of this thesis does not impose any theoretical reason why the force of mortality must be positive, and it is the negative values of the $\kappa^{(0)}$ parameters for females which make it a practical possibility that the force of mortality may become negative at the youngest ages far into the future. It is unlikely that these negative forces of mortality are a major barrier to the practical applicability of the model. For assurance business, the scenario where mortality falls to zero is unlikely to be of much interest as it is favourable for the insurer. For deferred annuity business, as mortality rates are so low around age 30, the value of a deferred annuity is unlikely to be significantly understated as a result of underestimating mortality at these ages. The business affected will, in any case, not come into force for some years.

5.4 Ages above the highest age in the data

In order to use these projections to calculate annuity values, it was necessary to calculate extrapolated mortality rates at ages above the highest age in the data. For the 2005 mortality curve, before the introduction of gamma parameters, the simplest approach is to perform this extrapolation log-linearly, with the slope and intercept of the line set so that the logarithm of the force of mortality is a differentiable function of age at the highest age of the data set. Specifically, let a be the highest age of the data set, so that a = 90 for CMI data and a = 89 for England and Wales data. It is assumed that the force of mortality in 2005 at ages $x > a$, before the introduction of gamma parameters, is:

$$\mu^0_{x,2005} = \mu^0_{a,2005} \exp\big(\lambda[x-a]\big),$$

where λ is calculated so as to make $\log \mu^0_{x,2005}$ a differentiable function of x at $x = a$. We find that:

$$\lambda = \frac{[\kappa^{(4)}_{2005} + 2\kappa^{(5)}_{2005}(a-\bar{x})]\exp[\kappa^{(3)}_{2005} + \kappa^{(4)}_{2005}(a-\bar{x}) + \kappa^{(5)}_{2005}((a-\bar{x})^2 - \hat{\sigma}^2_x)]}{\mu^0_{a,2005}}.$$

This extrapolation method, which we shall refer to as the 'log-linear extrapolation method', will be used in most of the applications in this chapter. Section A.3.6 of LifeMetrics (2007) suggests an alternative extrapolation method which we shall also investigate for comparison.

In this method, the 2005 force of mortality at ages above *a*, before the introduction of gamma parameters, is assumed to be a cubic polynomial function of age, with coefficients determined by the following four constraints. The log-linear extrapolation method satisfies the first two of these constraints but not (in general) the last two.

- The force of mortality is a continuous function of age *x* at *x* = *a*.
- The derivative of the force of mortality is a continuous function of age *x* at *x* = *a*.
- The force of mortality at age 119 is that corresponding to an initial mortality rate of 0.6, *i.e.* –log (1 – 0.6).
- The derivative of the force of mortality at age 119 is zero.

In this extrapolation method, all remaining lives are assumed to die just before their 120th birthday.

Figures 5.10 and 5.11 respectively show the extrapolations that would have been obtained if the log-linear and the LifeMetrics extrapolation method (respectively) had been applied in the past. Unlike Figure 5.6, the mortality shown in the region of the data in these graphs is the fitted mortality, before introducing gamma parameters, rather than the crude mortality.

The conclusion to be drawn from Figures 5.10 and 5.11 is that both extrapolation methods are potentially appropriate for extrapolating the 2005 mortality curve to ages above *a* but, for different reasons in each case, neither method is likely to be appropriate for extrapolating projected future mortality curves. In the case of the log-linear extrapolation method, there is some evidence of the historical extrapolated mortality curves being unstable from year to year, which is not a desirable feature to incorporate into a projection. This instability arises from the fact that small differences in mortality close to age *a* tend to be magnified by the extrapolation. In the case of the LifeMetrics extrapolation method, the method assumes that no improvements occur at age 119, which is an extreme assumption. Furthermore, even if this assumption is considered appropriate, then it is still an unnecessarily complex procedure to carry out an extrapolation using the LifeMetrics method for each future year in each scenario of a stochastic projection – it would be simpler to use the 'low improvement assumption' described below. Accordingly we shall not use either the log-linear or the LifeMetrics extrapolation method to extrapolate projected future mortality curves. We shall instead consider two possible assumptions as to how this extrapolation should be carried out, to demonstrate a range of potential outcomes:

- Mortality improvements after 2005 at ages above *a*, before the introduction of gamma parameters, occur at the same percentage rate as at age *a* in the same calendar year, *i.e.* for $x > a$ and $t > 2005$:

$$\mu^0_{xt} = \frac{\mu^0_{at}}{\mu^0_{a,2005}} \mu^0_{x,2005} .$$

  This assumption will be referred to as the 'high improvement assumption'.

- Mortality improvements after 2005 at ages close to and above *a*, before the introduction of gamma parameters, occur at a similar rate to those at age *a*. There are no mortality improvements after 2005 at age 120, before the introduction of gamma parameters. Improvement rates at ages between *a* and 120 are derived by linear interpolation. In other words, for $a < x \leq 120$ and $t > 2005$:

$$\mu^0_{xt} = \mu^0_{x,2005} \left[\frac{120 - x}{120 - a} \left(\frac{\mu^0_{at}}{\mu^0_{a,2005}} \right) + \frac{x - a}{120 - a} \right].$$

This assumption will be referred to as the 'low improvement assumption'.

The high improvement assumption produces lower projected forces of mortality at ages above *a*, and hence higher annuity values, than the low improvement assumption. The high improvement assumption might be considered appropriate if prudent annuity values are required and the low improvement assumption might be considered appropriate if best estimate annuity values are required.

Under the log-linear extrapolation method, no fixed limiting age was used at which all remaining lives were assumed to die. In practice the calculations of annuity values in Section 5.5 could only be performed up to a finite age, but it was verified in each case that allowing for the possibility of survival beyond that age would affect the calculated annuity value by less than 10^{-3}. In the case of the low improvement assumption, it was found in all cases that it was sufficient at this level of accuracy to consider only ages up to 120, so that there was no need to define a formula for the force of mortality at ages over 120 under the low improvement assumption.

5.5 Annuity values

Annuity values under each of the 500 scenarios were calculated for lives who had just attained each of the age labels 35, 45, 55, 65 and 75 at the end of 2005, based on all three data sets, under both the low and the high improvement assumptions. As a result of the age definitions used, *i.e.* age nearest birthday for CMI data and age last birthday for England and Wales data, this means that the exact ages of the lives at the end of 2005 were 34.5,44.5,…,74.5 for CMI data and 35,45,…,75 for England and Wales data, but for simplicity we shall refer to the ages only as 35,45,…,75 in 2005 in what follows. The annuities for lives aged 35, 45 and 55 in 2005 were deferred annuities payable from age label 65, *i.e.* from an exact age of 64.5 for CMI data and 65 for England and Wales data, and the annuities for lives aged 65 and 75 in 2005 were immediate annuities. The log-linear extrapolation method was used. A deterministic interest rate of 4.5% *p.a.* was used, and the annuities were assumed to be level single life annuities of 1 *p.a.* payable annually in advance. Thus the annuity value under the k^{th} scenario was calculated as:

$$\sum_{i=x_0-x}^{\infty} v^i S^k(x,i),$$

where:

$$S^k(x,i) = \exp\left(-\sum_{j=0}^{i-1} \mu^k_{x+j,2006+j}\right)$$

is the proportion of lives having just attained the age label x at the end of 2005 who survive for a further i years, $v = 1 / 1.045$, $x_0 = \max\{x,65\}$ and the superscript k represents the k^{th} scenario.

For CMI data, for lives aged 35 and 45 in 2005, the year of birth is outside the range where we identified our estimates of the gamma parameters as being reliable in Section 4.6. Accordingly, in these cases, instead of using the gamma parameter estimated from the data, it was taken from the stochastic projection. In all other cases, the gamma parameter estimated from the data was used.

Tables 5.2-5.14 show the mean, standard deviation and some key percentiles of the empirical distributions of the annuity values. Although not all the figures in these tables will be referred to in what follows, the same information is shown in all the tables for consistency. Table 5.1 shows which data set, which improvement assumption (high or low) and which extrapolation method were used in each of the tables 5.2-5.14. Table 5.1 also shows that, for one of the 12 possible combinations of data set, improvement assumption and extrapolation method, a

table was calculated using 5,000 scenarios in addition to the 500-scenario table. This was to investigate the extent to which it would be desirable to add more scenarios.

The following are the key observations from Tables 5.2-5.14:

- For all five values of the age in 2005, under both the low and the high improvement assumptions and under both the log-linear and the LifeMetrics extrapolation methods, the mean annuity value for England and Wales male data is less than that for England and Wales female data, which is less than that for CMI data. This ordering of the mean annuity values indicates that the mortality differential between assured lives and the general population is more significant than that between males and females.

- As the age in 2005 increases from x to $x + 10$, where $x = 35$, 45 or 55, *i.e.* before the annuities come into payment, we might expect the mean annuity value to increase by a factor close to $(1.045)^{10}$, as we are discounting for 10 fewer years. The fact that lives aged x in 2005 benefit from 10 years of mortality improvements compared with lives aged $x + 10$ in 2005 would tend to make the factor less than this, whereas the possibility that a life aged x in 2005 will not survive to age $x + 10$ would tend to make the factor greater than this. In fact, we observe that the factor is less than $(1.045)^{10}$ in all cases for an increase in age in 2005 from 35 to 45 or from 45 to 55. From 55 to 65, the factor is greater than $(1.045)^{10}$ except for CMI data under the high improvement assumption. This seems reasonable as there is a relatively significant probability of a life aged 55 dying before age 65. Only in the CMI data set under the high improvement assumption, with relatively high improvements over the period of the data, a relatively small probability of death between ages 55 and 65 and an equally high percentage rate of improvement above age 90 as at age 90, do the improvements outweigh the possibility of death between ages 55 and 65 to give a value of the factor less than $(1.045)^{10}$.

- As we move from the low to the high improvement assumption, it is clear that the mean annuity value should increase. The standard deviation of the annuity values, as a percentage of the mean, also increases. The justification for this is that, under the low improvement assumption, the extent of uncertainty about future mortality falls after age 90 (89 for England and Wales data), reducing to zero at age 120, whereas under the high improvement assumption, the uncertainty about future mortality, in percentage terms, is the same above age 90 as at age 90.

- The standard deviation of the annuity values, as a percentage of the mean, tends to be greater for younger than for older generations. This is not surprising in view of the remark made in Section 5.2 that under this model, uncertainty about the overall level of mortality increases significantly the further into the future projections are made (though uncertainty about the shape of the mortality curve does not increase significantly). The movement between ages 65 and 75 in 2005 is an exception to this. The increase in the standard deviation as a percentage of the mean from age 65 to 75 in 2005 is a consequence of the removal of the first ten years' annuity payments, which have a relatively high probability of being made under all scenarios.

- In terms of impact on the annuity values, the difference between the high and the low improvement assumptions is most significant (by far) for CMI data and least significant for England and Wales female data. As the CMI data set has generally lower base mortality and higher improvements than the England and Wales data sets, a higher proportion of lives survive to an age where the difference between the high and low improvement assumptions becomes relevant. The relatively high improvements also make the differences between projected mortality rates under the high and low improvement assumptions relatively large for CMI data. As the England and Wales female data set has generally lower improvements than the male data sets, the difference between projected mortality rates under the high and low improvement assumptions is less significant.

- Compared with the log-linear extrapolation method, the LifeMetrics extrapolation method produces lower mean annuity values for CMI data and higher mean annuity values for both England and Wales data sets. The differences are smaller than the differences between the low and high improvement assumptions. To gain an understanding of why the change of extrapolation method has the opposite effect for CMI data to that for England and Wales data, the graphs in Figure 5.12 were plotted. In these graphs, the solid curves, representing the extrapolated 2005 mortality curves under the log-linear extrapolation method, before the introduction of gamma parameters, are straight lines above age 90 (89 for England and Wales data). The dashed curves represent the LifeMetrics extrapolation method. We see that for the England and Wales data sets, mortality is lower under the LifeMetrics extrapolation method than under the log-linear extrapolation method at all ages above 89, which explains why the LifeMetrics annuity values are higher. For CMI data, the LifeMetrics extrapolation method produces slightly higher forces of mortality than the log-linear method from age 90 up to and including age 103, but significantly lower forces of mortality above age 103. The proportion of lives surviving to age 103 is sufficiently small that the higher forces of mortality from age 90 to age 103 outweigh the lower forces of mortality above age 103. Thus the LifeMetrics annuity values are lower than the log-linear annuity values.

- A comparison of the mean and median annuity values in Table 5.8 with those in Table 5.5 reveals that if we are only interested in the mean or median annuity values, then there is little benefit to be gained from considering more than 500 scenarios as the sampling errors incurred as a result will generally be smaller than the impact of the choice of extrapolation method, and much smaller than the impact of the choice of improvement assumption. If we are interested in relatively extreme percentiles of the distributions of annuity values, then it becomes more important to use a large number of scenarios. In this thesis, although we shall make some reference to percentiles of the distributions, including some relatively extreme percentiles, we are primarily interested in mean annuity values, with Section 5.6 focusing on the distribution of the conditional mean annuity value after one year.

The annuity values we have calculated will now be compared with values from the same previously published projections as referred to in Section 5.3. The values from the previously published projections are shown in Tables 5.15-5.17, and were calculated consistently with the values in Tables 5.2-5.8. In particular, the log-linear extrapolation method was used. We shall discuss the results in Tables 5.15-5.17 for ages 35-65 in 2005 first, and then discuss the results for age 75 in 2005.

For lives aged 65 in 2005 in CMI data, the model of this thesis generally predicts higher annuity values, *i.e.* greater mortality improvements, than the previously published projections. The mean annuity value of 14.305 under the high improvement assumption is higher than all the values from the previously published projections, and the mean value of 14.160 under the low improvement assumption is only slightly lower than the value of 14.182 from the Long Cohort projection. Under the low improvement assumption, over 90% of the annuity values exceed that under the Medium Cohort projection, just under 90% exceed that under the Medium Cohort projection subject to a 1% underpin, and just under 50% exceed that under the Long Cohort projection. Under the high improvement assumption, just under 95% of the annuity values exceed that under the Medium Cohort projection, between 90% and 95% exceed that under the Medium Cohort projection subject to a 1% underpin and between 60% and 70% exceed that under the Long Cohort projection.

For younger generations in CMI data, the annuity values from the previously published projections move further towards the lower end of the ranges of annuity values from the model of this thesis. In the extreme, for lives aged 35 in 2005, all but two of the 500 annuity values under the low improvement assumption exceed even that from the Long Cohort projection and all 500 annuity values under the high improvement assumption exceed even that from the Long Cohort projection.

For lives aged 65 in 2005 in England and Wales male data, the model of this thesis predicts generally lower annuity values, *i.e.* lower mortality improvements, than the GAD projections. The mean annuity values of 12.546 under the low improvement assumption and 12.591 under the high improvement assumption are lower than all the values from the GAD projections. The annuity values from the GAD projections do, however, lie within the ranges produced by the model of this thesis. Between 30% and 40% of the annuity values under the low improvement assumption, and between 40% and 50% under the high improvement assumption, exceed that from the LLE projection. Between 5% and 10% of the annuity values under the low improvement assumption, and between 10% and 20% of the annuity values under the high improvement assumption, exceed that from the Principal projection. Between 0.5% and 1% of the annuity values under the low improvement assumption, and just under 2.5% of the annuity values under the high improvement assumption, exceed that from the HLE projection.

For younger generations in England and Wales male data, the model of this thesis gives annuity values more in line with those from the GAD projections. For lives aged 55 in 2005, the mean annuity value under both the low and high improvement assumptions lies between the values from the LLE and Principal projections. For lives aged 45 in 2005, the mean annuity value under the low improvement assumption lies between the values from the LLE and Principal projections, while the mean annuity value under the high improvement assumption lies between the values from the Principal and HLE projections. For lives aged 35 in 2005, the mean annuity value under both the low and high improvement assumptions lies between the values from the Principal and HLE projections. However, one notable feature of Table 5.16 compared with Tables 5.4 and 5.5 is that the difference between the LLE, Principal and HLE projections is wider compared with the range of annuity values from the model of this thesis for younger generations than for older generations. The explanation for this is that the range of annuity values from the model of this thesis represents the potential impact of stochastic fluctuations, which have some tendency to average out over long periods, whereas the difference between the LLE, Principal and HLE projections represents the potential impact of *parameter risk*, *i.e.* the risk that parameters have been estimated incorrectly. As mentioned in Section 5.3, the LLE, Principal and HLE projections all assume that improvement rates tend towards a common value across all ages between 2004 (the base year of the projections) and 2029, the difference between the projections being that this common value is set to zero for the LLE projection, 1% for the Principal projection and 2% for the HLE projection. It would be a valuable topic for further research to produce stochastic projections that allow for both stochastic fluctuations and parameter risk together.

For England and Wales female data, the model of this thesis again generally predicts lower annuity values than the GAD projections. For females aged 65 in 2005, the mean annuity values under both the low and high improvement assumptions are less than the values from the GAD projections. This continues to be the case for females aged 55 in 2005, but for females aged 35 and 45 in 2005, the mean annuity value is between the LLE projection and the Principal projection. For females aged 65 in 2005, the annuity value from the Principal projection is between the 95^{th} and 97.5^{th} percentiles of the distribution of annuity values from the model of this thesis under the low improvement assumption and between the 90^{th} and 95^{th} percentiles under the high improvement assumption. As the age in 2005 decreases, the Principal annuity value moves rather closer to the centre of the distribution, being close to the 80^{th} percentile under the high improvement assumption, and between the 80^{th} and 90^{th} percentiles under the low improvement assumption, for age 35 in 2005. As for England and Wales male data, the difference between the LLE, Principal and HLE projections is wider compared with the range of annuity values from the model of this thesis for younger than for older generations.

As the age in 2005 increases from 65 to 75, the trend for CMI data for the annuity values from the model of this thesis to decrease relative to those from the previously published projections with increasing age continues, but under the Medium Cohort projection and the Medium Cohort projection subject to a 1% underpin, it remains the case that the mean annuity value from the previously published projection is less than from the model of this thesis, under both the low and high improvement assumptions. For England and Wales data, however, the trend reverses and the annuity values from the previously published projections move closer to the centre of the distribution from the model of this thesis. Despite this, the values from the

previously published projections remain above the mean values from the model of this thesis, with the exception of the high improvement assumption compared with the LLE projection for male data.

To gain a greater understanding of the reasons for the differences between Tables 5.2-5.7 and Tables 5.15-5.17, Figures 5.13-5.24 were plotted. Figures 5.13-5.18 compare the distribution of the projected forces of mortality at each future age with the previously published projections, and are similar to Figures 5.7-5.9 except that we are following a particular generation as it ages rather than considering what mortality rate each generation experiences as it passes a particular age. Figures 5.19-5.24 compare the distribution of the projected number of survivors to each future age with the previously published projections.

The following are the key observations from Figures 5.13-5.18:

- For CMI data, most of the stochastic scenarios produce higher forces of mortality in the first year of the projection, for the ages considered, than the previously published projections. Thus the improvement rates from over 2006 predicted by the model of this thesis are generally lower than those shown in the previously published projections.

- For England and Wales data, for both males and females, most of the stochastic scenarios again produce higher forces of mortality in the first year of the projection than the previously published projections, although there are exceptions. This again indicates generally lower 2006 improvement rates predicted by the model of this thesis than by the previously published projections.

- All the graphs for CMI data show that the Medium Cohort projection, the Medium Cohort projection subject to a 1% underpin and the Long Cohort projection are identical below age 65. The reason for this is that by 2005, the range of years of birth for which the Medium Cohort projection is different from the projection in CMI (1999) has narrowed to 1914-1938, and the range of years of birth for which the Long Cohort projection is different has narrowed to 1912-1940. Lives born in 1940 are aged 65 in 2005. In addition, for lives in the age range considered in 2005, the 1% underpin does not 'bite' until after the lives have reached age 65.

- For England and Wales data, for both males and females, the differences between the three GAD projections in the early years are smaller than the range of the stochastic scenarios. This is because the differences between the GAD projections relate to the long-term value improvement rates tend towards, whereas the range of the stochastic scenarios relates to random fluctuations.

- For CMI data, for lives aged 35, 45 and 55 in 2005, most of the stochastic scenarios continue to produce higher forces of mortality than the previously published projections until the life passes age 60. In the model of this thesis, mortality at these younger ages is driven mainly by the $\kappa^{(0)}$ parameters, which are assumed to revert to a mean level, and so improvement rates are lower below age 60 than at age 60. However, in the previously published projections, as defined in the 'library', future mortality improvement rates below age 60 are assumed to be the same as at age 60.

- For England and Wales data, for both males and females, most of the scenarios produce higher forces of mortality over the age range 60-80 than the Principal GAD projection. This effect is generally stronger for females than for males. The differences tend to be larger for higher values of the age in 2005, although the reverse is the case as the age in 2005 increases from 65 to 75.

- For CMI data, as the age approaches 90, mortality from the model of this thesis decreases relative to mortality from the previously published projections. The previously published projections exhibit mortality improvements which slow down as the age approaches 90. The model of this thesis, on the other hand, forecasts the same long-term percentage rate of improvement at all ages, because the

improvements are driven entirely by the downward trend in the $\kappa^{(3)}$ parameters. (An exception to this is at the youngest ages, where the positive $\kappa^{(0)}$ parameters cause mortality to improve more slowly than this.)

- At the oldest ages, under the low improvement assumption, the model of this thesis produces higher forces of mortality than any previously published projection under which the improvement rates do not approach zero as the age approaches 120. For CMI data, this applies to the Medium Cohort projection subject to a 1% underpin. For both the England and Wales data sets, this applies to the Principal projection and to the HLE projection.

- At the oldest ages under the high improvement assumption for CMI data, all the stochastic scenarios produce lower forces of mortality than the Medium Cohort and Long Cohort projections, as mortality improvements under the model of this thesis do not approach zero at the oldest ages but those under the Medium Cohort and Long Cohort projections do. There are only a few scenarios producing higher forces of mortality than the Medium Cohort projection subject to a 1% underpin, *i.e.* there are only a few scenarios for which the model of this thesis predicts improvements of less than 1% *p.a.* at the oldest ages.

- At the oldest ages under the high improvement assumption for England and Wales data, most or all of the stochastic scenarios produce higher forces of mortality than both the Principal and HLE projections, the former representing a long-term improvement rate of 1% *p.a.* and the latter representing a long-term improvement rate of 2% *p.a.* Most or all of the scenarios produce lower forces of mortality than the LLE projection, as the improvement rates under the scenarios do not approach zero as the age approaches 120 but those under the LLE projection do.

- In most of the graphs, the interval between the 2.5th and 97.5th percentiles widens with increasing age up to age 90 (89 for England and Wales data), as a result of the expanding funnels of doubt in Figures 5.1-5.5.

- In the graphs for age 35 in 2005 for CMI data (Figures 5.13(i) and 5.14(i)), the interval between the 2.5th and 97.5th percentiles stands out as being particularly wide in the early years of the projection. This is due to two main factors. Firstly, the gamma parameter comes from the projection rather than from the data and is therefore subject to stochastic variation – the only other graphs where the gamma parameter comes from the projection are Figures 5.13(ii) and 5.14(ii). Secondly, the uncertainty in the $\kappa^{(0)}$ parameters gives rise to significant uncertainty in log mortality at younger ages.

- As the age increases above 90 (89 for England and Wales data), the width of the interval between the 2.5th and 97.5th percentiles under the high improvement assumption remains approximately constant. There is a slight increase arising from the continued expansion of the funnel of doubt in the $\kappa^{(3)}$ parameters, but the funnels of doubt in the other parameters have almost reached their ultimate width by this point. Under the low improvement assumption, the width of the interval decreases, reaching zero at age 120 where the projected force of mortality under all scenarios is the same as that extrapolated from the 2005 mortality curve.

The vertical position of a point in Figures 5.19-5.24 is determined by the vertical positions of all the points to the left of it in Figures 5.13-5.18, *i.e.* the proportion of lives surviving to a given age is determined by mortality rates at all younger ages. Thus Figures 5.19-5.24 contain the same information as Figures 5.13-5.18, with the exception that the range of the stochastic scenarios is rather narrower in Figures 5.19-5.24 because of scenarios that have high mortality in some future years but low mortality in others. The main reason why the presentation of Figures 5.19-5.24 is useful is to give an understanding of how the different projected forces of mortality in Figures 5.13-5.18 affect annuity values. The areas under the graphs in Figures 5.19-5.24 are expectations of life, *i.e.* annuity values calculated at a 0% interest rate.

5.6 Individual Capital Assessments (ICAs)

We shall now apply the model to risk-based capital calculations under the Individual Capital Assessment (ICA) regime of the UK insurance regulator, the Financial Services Authority (FSA), for deferred and immediate annuities. We shall continue to use a deterministic interest rate of 4.5%. Mortality at ages above 90 (89 for England and Wales data) will be assumed to be in accordance with the log-linear extrapolation method and the high improvement assumption. For lives aged 35 and 45 in 2005 in CMI data, we shall not use the stochastic projection of the gamma parameters, but we shall instead use a deterministic projection with all the future innovation terms set to zero, as used for the previously published projections in Section 5.3.

Section 7.1.42R of the FSA's *Prudential Sourcebook for Insurers* (INSPRU) states:

'Where the FSA requests a firm to submit to it a written record of the firm's assessments of the adequacy of its capital resources…those assessments must include an assessment comparable to a 99.5% confidence level over a one year timeframe that the value of assets exceeds the value of liabilities, whether or not this is the confidence level otherwise used in the firm's own assessments.'

If the calculation described is performed by a simulation approach, then it requires a number of scenarios to be generated for the kappa parameter values, and the resulting mortality experience, in 2006 (the first year beyond the period of the data). For lives that survive 2006, the mean annuity value must then be calculated separately under each scenario. These mean annuity values need to be multiplied by the probability of surviving 2006 in the same scenario, and the 99.5th percentile of the resulting distribution needs to be taken. As the mean annuity values themselves need to be calculated by a simulation approach, we are faced with having to run simulations within simulations, which is highly computationally intensive. Accordingly we shall develop a less computationally intensive approximation to the ICA, and carry out some testing to confirm that the approximation is sufficiently accurate.

Consider a group of N deferred or immediate annuitants who have just attained the age label x at the start of calendar year t, where calendar year $t-1$ is the last year of the data. (The assumption that the annuitants have just attained a particular age label is not critical to the analysis, but has been made to simplify the presentation.) The annuities are assumed to be payable annually in advance. x_1 denotes the age label from which the annuities are payable for deferred annuities, and is set equal to $x+1$ for immediate annuities. The probability of a particular one of the annuitants surviving to age label $x+1$ is:

$$\exp(-\mu_{xt}) = \exp(-\gamma_{t-x}[\kappa_t^{(0)} + \exp\{\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)\}]).$$

The expected number of annuitants surviving to age label $x+1$ is therefore:

$$N\exp(-\gamma_{t-x}[\kappa_t^{(0)} + \exp\{\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)\}]).$$

In what follows, we shall assume that N is sufficiently large that the actual number of annuitants surviving to age label $x+1$ may be assumed to be equal to the above value.

We now use the time series models fitted in Chapter 4 to express the kappa parameters for year t in terms of the innovations for year t and other quantities that are assumed to be known:

$$\kappa_t^{(i)} = \mu^{(i)} + \alpha^{(i)}(\kappa_{t-1}^{(i)} - \mu^{(i)}) + \sigma^{(i)}Z_t^{(i)}, i = 0,4,5,$$
$$\kappa_t^{(3)} = \kappa_{t-1}^{(3)} + \mu^{(3)} + \sigma^{(3)}Z_t^{(3)} + \sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)},$$

where $Z_t^{(0)}$, $Z_t^{(3)}$, $Z_t^{(4)}$ and $Z_t^{(5)}$ are jointly normally distributed with zero mean, unit variance and the correlations shown in Section 4.7. A small letter *z* has been used for the innovation term $z_{t-1}^{(3)}$ from year *t* – 1 to signify that the value is assumed known.

The number of annuitants surviving to age label *x* + 1 is then:

$$N\exp\left(-\gamma_{t-x}\left[\begin{array}{l}\mu^{(0)}+\alpha^{(0)}(\kappa_{t-1}^{(0)}-\mu^{(0)})+\sigma^{(0)}Z_t^{(0)} \\ +\exp\left\{\begin{array}{l}\kappa_{t-1}^{(3)}+\mu^{(3)}+\sigma^{(3)}Z_t^{(3)}+\sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)} \\ +[\mu^{(4)}+\alpha^{(4)}(\kappa_{t-1}^{(4)}-\mu^{(4)})+\sigma^{(4)}Z_t^{(4)}](x-\bar{x}) \\ +[\mu^{(5)}+\alpha^{(5)}(\kappa_{t-1}^{(5)}-\mu^{(5)})+\sigma^{(5)}Z_t^{(5)}]((x-\bar{x})^2-\hat{\sigma}_x^2)\end{array}\right\}\end{array}\right]\right).$$

We now linearise this expression in the $Z_t^{(l)}$. We shall refer to this linearisation as Approximation 1 in what follows. We shall later carry out some testing to confirm that the error introduced by the linearisation is acceptably small. We find that the number of annuitants surviving to age label *x* + 1 is:

$$N\overline{p}_{xt}[1+a_0Z_t^{(0)}+a_3Z_t^{(3)}+a_4Z_t^{(4)}+a_5Z_t^{(5)}],$$

where:

$$\begin{aligned}
a_0 &= -\gamma_{t-x}\sigma^{(0)}, \\
a_3 &= -\gamma_{t-x}\sigma^{(3)}\overline{\mu}_{\exp}(x,t), \\
a_4 &= -\gamma_{t-x}\sigma^{(4)}(x-\bar{x})\overline{\mu}_{\exp}(x,t), \\
a_5 &= -\gamma_{t-x}\sigma^{(5)}((x-\bar{x})^2-\hat{\sigma}_x^2)\overline{\mu}_{\exp}(x,t), \\
\overline{\kappa}_t^{(i)} &= \mu^{(i)}+\alpha^{(i)}(\kappa_{t-1}^{(i)}-\mu^{(i)}), i=0,4,5, \\
\overline{\kappa}_t^{(3)} &= \kappa_{t-1}^{(3)}+\mu^{(3)}+\sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)}, \\
\overline{\mu}_{\exp}(x,t) &= \exp\{\overline{\kappa}_t^{(3)}+\overline{\kappa}_t^{(4)}(x-\bar{x})+\overline{\kappa}_t^{(5)}((x-\bar{x})^2-\hat{\sigma}_x^2)\}, \\
\overline{\mu}_{xt} &= \gamma_{t-x}[\overline{\kappa}_t^{(0)}+\overline{\mu}_{\exp}(x,t)], \\
\overline{p}_{xt} &= \exp(-\overline{\mu}_{xt}).
\end{aligned}$$

Consider an annuitant who has survived to age label *x* + 1. We express the (random) value at the start of year *t* + 1 of the liabilities in respect of this annuitant, *L(t* + 1), in terms of the $Z_t^{(l)}$, some 'future annuity values' independent of the $Z_t^{(l)}$, some of which have payments varying over time and/or are calculated at revised interest rates, and other quantities that are assumed to be known. In deriving this expression, we again linearise in the $Z_t^{(l)}$, and we shall refer to this linearisation as Approximation 2 in what follows. Appendix D shows that the resulting expression is:

$$L(t+1)=FutAnnVal(x,t)+b_0Z_t^{(0)}+b_3Z_t^{(3)}+b_4Z_t^{(4)}+b_5Z_t^{(5)},$$

where:

$$b_0 = \frac{\gamma_{t-x}\sigma^{(0)}(\alpha^{(0)})^{a-x+1}(e^{\lambda}-1)}{(1-\alpha^{(0)})(1-\alpha^{(0)}e^{\lambda})}\,FutAnnVal\_Maxa(x,t)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}}{1-\alpha^{(0)}}\,FutAnnVal\_Maxa\_alpha0(x,t)$$

$$-\gamma_{t-x}\sigma^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right)FutAnnVal(x,t)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}}{1-\alpha^{(0)}e^{\lambda}}\,FutAnnVal\_Mina\_alpha0\_lambda(x,t),$$

$$b_3 = -\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})FutAnnVal\_Varying3(x,t),$$

$$b_4 = -\gamma_{t-x}\sigma^{(4)}FutAnnVal\_Varying4(x,t),$$

$$b_5 = -\gamma_{t-x}\sigma^{(5)}FutAnnVal\_Varying5(x,t),$$

$$FutAnnVal(x,t) = \sum_{i=x_1-x-1}^{\infty} v^i \exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right),$$

$$FutAnnVal\_Maxa(x,t) = \sum_{i=x_1-x-1}^{a-x-1} v^i \exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right),$$

$$FutAnnVal\_Maxa\_alpha0(x,t) = \sum_{i=x_1-x-1}^{a-x-1} (v\alpha^{(0)})^i \exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$FutAnnVal\_Mina\_alpha0\_lambda(x,t) = \sum_{i=a-x}^{\infty} (v\alpha^{(0)}e^{\lambda})^i \exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right).$$

$$FutAnnVal\_Varying3(x,t) = \sum_{i=x_1-x-1}^{\infty} v^i\left(\sum_{j=1}^{i}\overline{\mu}_{\exp}(x+j,t+j)\right)\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right),$$

$$FutAnnVal\_Varying4(x,t) = \sum_{i=x_1-x-1}^{a-x-1} \frac{v^i\left(\sum_{j=1}^{i}(\alpha^{(4)})^j(x+j-\bar{x})\overline{\mu}_{\exp}(x+j,t+j)\right)}{\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)}$$

$$+\sum_{i=a-x}^{\infty} \frac{v^i\left(\sum_{j=1}^{a-x-1}(\alpha^{(4)})^j(x+j-\bar{x})\overline{\mu}_{\exp}(x+j,t+j)+(a-\bar{x})\sum_{j=a-x}^{i}(\alpha^{(4)})^j\overline{\mu}_{\exp}(x+j,t+j)\right)}{\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)},$$

$$FutAnnVal\_Varying5(x,t) = \sum_{i=x_1-x-1}^{a-x-1} \frac{v^i\left(\sum_{j=1}^{i}(\alpha^{(5)})^j((x+j-\bar{x})^2-\hat{\sigma}_x^2)\overline{\mu}_{\exp}(x+j,t+j)\right)}{\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)}+$$

$$+\sum_{i=a-x}^{\infty} v^i \begin{pmatrix} \sum_{j=1}^{a-x-1} (\alpha^{(5)})^j ((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\bar{\mu}_{\exp}(x+j,t+j) \\ +((a-\bar{x})^2 - \hat{\sigma}_x^2) \sum_{j=a-x}^{i} (\alpha^{(5)})^j \bar{\mu}_{\exp}(x+j,t+j) \end{pmatrix}$$
$$\exp\left(-\sum_{j=1}^{i} \bar{\mu}_{x+j,t+j}\right)$$

and λ is as in Section 5.4.

Multiplying this by the number of survivors to age label $x + 1$ gives the (random) value at the start of year $t + 1$ of all the liabilities together as:

$$N\bar{p}_{xt}[1-\gamma_{t-x}\sigma^{(0)}Z_t^{(0)} - \gamma_{t-x}\sigma^{(3)}\bar{\mu}_{\exp}(x,t)Z_t^{(3)} - \gamma_{t-x}\sigma^{(4)}(x-\bar{x})\bar{\mu}_{\exp}(x,t)Z_t^{(4)}$$
$$-\gamma_{t-x}\sigma^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)\bar{\mu}_{\exp}(x,t)Z_t^{(5)}]$$
$$\left[\begin{array}{l} FutAnnVal(x,t) + \dfrac{\gamma_{t-x}\sigma^{(0)}(\alpha^{(0)})^{a-x+1}(e^{\lambda}-1)Z_t^{(0)}}{(1-\alpha^{(0)})(1-\alpha^{(0)}e^{\lambda})} FutAnnVal\_Maxa(x,t) \\ + \dfrac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}Z_t^{(0)}}{1-\alpha^{(0)}} FutAnnVal\_Maxa\_alpha0(x,t) \\ - \gamma_{t-x}\sigma^{(0)}Z_t^{(0)}\left(\dfrac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}} + \dfrac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right) FutAnnVal(x,t) \\ + \dfrac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}Z_t^{(0)}}{1-\alpha^{(0)}e^{\lambda}} FutAnnVal\_Mina\_alpha0\_lambda(x,t) \\ - \gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})Z_t^{(3)} FutAnnVal\_Varying3(x,t) \\ - \gamma_{t-x}\sigma^{(4)}Z_t^{(4)} FutAnnVal\_Varying4(x,t) - \gamma_{t-x}\sigma^{(5)}Z_t^{(5)} FutAnnVal\_Varying5(x,t) \end{array}\right]$$
$$= N\bar{p}_{xt}\left[c_0 Z_t^{(0)} + c_3 Z_t^{(3)} + c_4 Z_t^{(4)} + c_5 Z_t^{(5)}\right]$$

after linearisation in the $Z_t^{(l)}$, where:

$$c_0 = \frac{\gamma_{t-x}\sigma^{(0)}(\alpha^{(0)})^{a-x+1}(e^{\lambda}-1)}{(1-\alpha^{(0)})(1-\alpha^{(0)}e^{\lambda})} FutAnnVal\_Maxa(x,t)$$
$$+ \frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}}{1-\alpha^{(0)}} FutAnnVal\_Maxa\_alpha0(x,t)$$
$$- \gamma_{t-x}\sigma^{(0)}\left(1 + \frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}} + \frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right) FutAnnVal(x,t)$$
$$+ \frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}}{1-\alpha^{(0)}e^{\lambda}} FutAnnVal\_Mina\_alpha0\_lambda(x,t),$$
$$c_3 = -\gamma_{t-x}\sigma^{(3)}\bar{\mu}_{\exp}(x,t) FutAnnVal(x,t)$$
$$- \gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)}) FutAnnVal\_Varying3(x,t),$$

$$
\begin{aligned}
c_4 &= -\gamma_{t-x}\sigma^{(4)}(x-\bar{x})\bar{\mu}_{\exp}(x,t)FutAnnVal(x,t) \\
&- \gamma_{t-x}\sigma^{(4)}FutAnnVal\_Varying4(x,t), \\
c_5 &= -\gamma_{t-x}\sigma^{(5)}((x-\bar{x})^2-\hat{\sigma}_x^2)\bar{\mu}_{\exp}(x,t)FutAnnVal(x,t) \\
&- \gamma_{t-x}\sigma^{(5)}FutAnnVal\_Varying5(x,t).
\end{aligned}
$$

This linearisation will be referred to as Approximation 3 in what follows.

As the 'future annuity values', *e.g.* $FutAnnVal(x,t)$, do not depend on the $Z_t^{(l)}$, the conditional expected value at the start of year *t* + 1 of all the liabilities, given the $Z_t^{(l)}$, is therefore *NC*(*x*), where:

$$
C(x) = \bar{p}_{xt}\left[\begin{aligned}
&E[FutAnnVal(x,t)] \\
&+\frac{\gamma_{t-x}\sigma^{(0)}(\alpha^{(0)})^{a-x+1}(e^{\lambda}-1)Z_t^{(0)}}{(1-\alpha^{(0)})(1-\alpha^{(0)}e^{\lambda})}E[FutAnnVal\_Maxa(x,t)] \\
&+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}Z_t^{(0)}}{1-\alpha^{(0)}}E[FutAnnVal\_Maxa\_alpha0(x,t)] \\
&-\gamma_{t-x}\sigma^{(0)}Z_t^{(0)}E[FutAnnVal(x,t)] \\
&-\gamma_{t-x}\sigma^{(0)}Z_t^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}+\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right)E[FutAnnVal(x,t)] \\
&+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}Z_t^{(0)}}{1-\alpha^{(0)}e^{\lambda}}E[FutAnnVal\_Mina\_alpha0\_lambda(x,t)] \\
&-\gamma_{t-x}\sigma^{(3)}\bar{\mu}_{\exp}(x,t)Z_t^{(3)}E[FutAnnVal(x,t)] \\
&-\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})Z_t^{(3)}E[FutAnnVal\_Varying3(x,t)] \\
&-\gamma_{t-x}\sigma^{(4)}(x-\bar{x})\bar{\mu}_{\exp}(x,t)Z_t^{(4)}E[FutAnnVal(x,t)] \\
&-\gamma_{t-x}\sigma^{(4)}Z_t^{(4)}E[FutAnnVal\_Varying4(x,t)] \\
&-\gamma_{t-x}\sigma^{(5)}((x-\bar{x})^2-\hat{\sigma}_x^2)\bar{\mu}_{\exp}(x,t)Z_t^{(5)}E[FutAnnVal(x,t)] \\
&-\gamma_{t-x}\sigma^{(5)}Z_t^{(5)}E[FutAnnVal\_Varying5(x,t)]
\end{aligned}\right]
$$

and all the expectations can be calculated in the same way as the mean annuity values in Section 5.5, except that some of the annuities have payments varying over time or are valued at revised interest rates, The 'annuitant' has just attained the age label *x* + 1 rather than *x*, and the starting kappa parameters have been rolled forward for one year on best estimate assumptions. Note that the random variable *C*(*x*) is a constant plus a linear combination of the $Z_t^{(l)}$.

It is instructive to consider how the different terms in the above expression for *C*(*x*) arise. $\bar{p}_{xt}E[FutAnnVal(x,t)]$ is the expected value at the start of year *t* + 1 of all the liabilities conditional on all the $Z_t^{(l)}$ being equal to zero. All the other terms containing $E[FutAnnVal(x,t)]$, with the exception of $-\gamma_{t-x}\sigma^{(0)}Z_t^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}+\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right)E[FutAnnVal(x,t)]$, arise from the

variation of the number of lives surviving year *t* with the $Z_t^{(l)}$. The remaining terms arise from the variation of the conditional mean annuity value at the end of year *t* with the $Z_t^{(l)}$. Thus for each kappa parameter, there is one term inside the square brackets arising from the variation of the number of lives surviving year *t* and one term arising from the variation of the conditional mean annuity value, with the exception that there are four different terms containing $Z_t^{(0)}$ arising from the variation of the conditional mean annuity value. The sum of these four terms can alternatively be written as:

$$
\begin{aligned}
&-\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}Z_t^{(0)}}{1-\alpha^{(0)}}E[FutAnnVal\_Maxa(x,t)] \\
&+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}Z_t^{(0)}}{1-\alpha^{(0)}}E[FutAnnVal\_Maxa\_alpha0(x,t)] \\
&-\gamma_{t-x}\sigma^{(0)}Z_t^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}+\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right) \\
&(E[FutAnnVal(x,t)]-E[FutAnnVal\_Maxa(x,t)]) \\
&+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}Z_t^{(0)}}{1-\alpha^{(0)}e^{\lambda}}E[FutAnnVal\_Mina\_alpha0\_lambda(x,t)].
\end{aligned}
$$

The first and third of these terms can be thought of as the impact of $Z_t^{(0)}$ on the conditional mean value of the annuity payments up to and including age *a* and after age *a* respectively, ignoring the fact that not all of the impact of $Z_t^{(0)}$ will have occurred by the time these annuity payments are made as the conditional mean of the AR(1) process for the $\kappa^{(0)}$ parameters following the disturbance $Z_t^{(0)}$ will not have reverted completely to $\mu^{(0)}$ by the time the payments are made. The second and fourth terms, respectively, then make an adjustment for this to the impact on the conditional mean value of the annuity payments up to and including age *a* and after age *a* respectively.

For a portfolio of annuities on lives of different ages, the conditional expected value at the start of year *t* + 1 of all the liabilities is:

$$\sum_x N(x)C(x),$$

where *N*(*x*) is the number of annuitants at each age *x*. This conditional expected value is again a constant plus a linear combination of the $Z_t^{(l)}$. We should expect the portfolio to produce a lower ICA capital requirement than the sum of the capital requirements for the individual ages, as a result of diversification benefits arising from the fact that mortality changes at different ages are not perfectly correlated.

As we are assuming that interest rates are constant, and as the amount of annuity payments made in the first year is deterministic, being equal to the number of immediate annuitants, the value of the assets at the end of the first year is deterministic. Thus the ICA capital is simply the difference between the 99.5th percentile and the mean of $\sum_x N(x)C(x)$, as the $Z_t^{(l)}$ vary, discounted for one year. To calculate this, we use the fact that $\sum_x N(x)C(x)$ is normally distributed, with the variance being derived from the parameter estimates of the time series models in Chapter 4, from the 'future annuity values' calculated in the same way as the

mean annuity values in Section 5.5, and from the correlations between the different $Z_t^{(l)}$ in Section 4.7. The ICA capital is then given by:

$$v\Phi^{-1}(0.995)\sqrt{\text{var}\left(\sum_x N(x)C(x)\right)},$$

where Φ is the N(0,1) distribution function.

Only 50 scenarios, rather than 500, were used to calculate the 'future annuity values'. The possibility of adding further scenarios was investigated but it was found that the results for ICA capital changed only in the third decimal place, *i.e.* the last decimal place quoted in Table 5.18, as a result.

Table 5.18 shows the results for ICA capital considering only one age x in 2005 at a time, *i.e.* $N(y)$ is set to zero for $y \neq x$. $N(x)$ is set to 1 – this is for presentational purposes only and does not imply that it is sensible to use this method to calculate the ICA capital for a single annuity.

It is not apparent what relationship these ICA capital amounts should bear to the distributions of annuity values derived in Section 5.5. The former are based on distributions of mean annuity values after one year as the new information arising during the year varies, whereas the latter are ranges of outcomes over the full outstanding lifetime of the annuitants. A 'rule of thumb' for calculating ICA capital for longevity risk that is widely used within the UK life insurance industry is to take the 95th percentile of the distribution of annuity values derived in Section 5.5. If the discounted mean term of a portfolio of annuity business is approximately 10 years, then this might be considered reasonable on the grounds that a 99.5% confidence level over a one-year timeframe is approximately equivalent to a (100 – 0.5 x 10) = 95% confidence level over a 10-year timeframe. However, Table 5.18 shows that the ICA capital amounts we have calculated correspond to much lower percentiles than the 95th percentile of the distributions of annuity values, particularly for younger generations and for CMI data. An insurance company might construe this as supporting the use of a lower percentile than the 95th percentile in its ICA calculation. This may in some circumstances be reasonable, but the company should bear in mind the following before deciding to use a lower percentile than the 95th percentile:

- The most appropriate percentile to use will depend on the age distribution of the company's annuity portfolio, as the percentages in Table 5.18 vary significantly by age in 2005. The company should also consider how its annuity portfolio compares with the populations underlying the three data sets.

- The percentages in Table 5.18 make no allowance for parameter risk, *i.e.* the risk that the parameters of the model have been estimated incorrectly.

- The percentages in Table 5.18 make no allowance for model risk, *i.e.* the risk that the model itself is incorrect.

We might expect the percentages in Table 5.18 to be lower for groups of annuitants with a long future life expectancy, as this will tend to make the risk over one year relatively small compared with the risk over the whole future lifetime of the annuitants. This is indeed what we observe – the percentages increase as the age in 2005 increases, they increase as we move from CMI data to England and Wales female data, and in most cases they increase as we move from England and Wales female data to England and Wales male data.

Table 5.19 shows the coefficients of the different $Z_{2006}^{(l)}$ in the expressions for $C(x)$ for each age x and for each data set. Large absolute values in this table indicate that the ICA capital is, in absolute terms, highly sensitive to that 2006 innovation term. The following observations can be made from Table 5.19:

- All the coefficients of $Z_{2006}^{(0)}$ and $Z_{2006}^{(3)}$ are negative. This is as expected because an increase in these innovation terms is always associated with an increase in mortality and hence with a decrease in annuity values.

- All the coefficients of $Z_{2006}^{(4)}$ are again negative. An increase in this innovation term is associated with a decrease in mortality at ages below the mean age of the data set (60 for CMI data and 59.5 for England and Wales data) and an increase at ages above the mean age of the data set. The negative coefficients show that the effect of ages above the mean age of the data set always outweighs the effect of ages below the mean age of the data set. This can be explained by the low mortality rates below the mean ages of all the data sets and by the fact that, as the $\alpha^{(4)}$ parameters are relatively close to 1, $Z_{2006}^{(4)}$ has a not insignificant impact on mortality at older ages for lives that are still below the mean age.

- The coefficients of $Z_{2006}^{(5)}$ are positive for lower values of the age in 2005 but have become negative by the time the age in 2005 increases to 65 (75 for England and Wales male data). For CMI data, an increase in $Z_{2006}^{(5)}$ leads to a decrease in mortality between ages 43 and 77 inclusive and to an increase in mortality outside this age range. For England and Wales data, an increase in $Z_{2006}^{(5)}$ leads to a decrease in mortality between ages 43 and 76 inclusive and to an increase in mortality outside this age range. The positive coefficients of $Z_{2006}^{(5)}$ at ages in 2005 up to and including 55 show that the age range 43-77 or 43-76 has the greatest impact. Once the age in 2005 has increased to 65, mortality above age 77 (76 for England and Wales data) has more of an impact. We find that this impact outweighs the impact of the age range 43-77 or 43-76, to give a negative coefficient of $Z_{2006}^{(5)}$, for CMI data and for England and Wales female data but not for England and Wales male data. This seems reasonable as the England and Wales male data set has the highest probability of death before age 77 for a life now aged 65. Once the age in 2005 has increased to 75, the coefficient of $Z_{2006}^{(5)}$ is driven almost entirely by mortality above age 77 (76 for England and Wales data), so it is negative for all data sets.

- The coefficient of $Z_{2006}^{(3)}$ has the largest magnitude except at ages 65 and 75 for CMI data and at age 75 for England and Wales female data. For lower values of the age x in 2005, the key influence on $C(x)$ will be from ages relatively close to the mean age of the data set, and the $\kappa^{(3)}$ parameters are the main driver of mortality in this age range. However, as the $\kappa^{(3)}$ parameters are, in the long term, the only driver of mortality improvements over time in the model, it is not surprising that the coefficient of $Z_{2006}^{(3)}$ continues to be significant for the higher values of the age in 2005.

- At ages 65 and 75 for CMI data and 75 for England and Wales female data, the coefficient of $Z_{2006}^{(4)}$ has the largest magnitude. The $\kappa^{(4)}$ parameters provide a first-order approximation to how quickly the logarithm of the force of mortality increases with age as the age rises above the mean age of the data set, so it would be expected that the coefficient of $Z_{2006}^{(4)}$ would become increasingly significant in magnitude as the age in 2005 increases. Although the $\kappa^{(4)}$ parameters also have a significant impact in percentage terms on the force of mortality at the younger ages of the data sets, there are relatively few deaths at those ages. The higher estimated value of $\sigma^{(4)}$ for CMI data than for England and Wales female data, and for England and Wales female data than for England and Wales male data, provides an explanation as to why the change in the coefficient with the highest magnitude from

$Z_{2006}^{(3)}$ to $Z_{2006}^{(4)}$ occurs at a younger age for CMI data than for England and Wales female data, and at a younger age for England and Wales female data than for England and Wales male data.

- The $\kappa^{(0)}$ parameters have a relatively minor impact in most cases, with the coefficients of $Z_{2006}^{(0)}$ being much smaller in magnitude than those of $Z_{2006}^{(3)}$, even for the lower values of the age in 2005.

- The $\kappa^{(5)}$ parameters also have a relatively minor impact for ages in 2005 up to and including 65. However, for lives aged 75 in 2005, the coefficient of $Z_{2006}^{(5)}$ is significant in magnitude compared with those of $Z_{2006}^{(3)}$ and $Z_{2006}^{(4)}$.

Figures 5.25-5.27 provide a check on the ICA capital amounts in Table 5.18 and are also of interest in their own right. The figures show the projected future sizes of a fund set up in 2005 equal to the mean annuity value plus the ICA capital, for different values of the age x in 2005 and for each of the three data sets. As before, $N(x)$ has been set to 1 for presentational purposes only and $N(y)$ has been set to zero for $y \neq x$. The red curves show the $(100 - p)^{th}$ percentiles of the distributions of the future fund size, where p% is the percentile specified in Table 5.18. It is the fact that the red curves reach a fund size close to zero at age 120 that provides the check on the ICA capital amounts. The following remarks can also be made on Figures 5.25-5.27:

- At ages under 65, the graphs show no stochastic variation as no annuity payments are being made. The fund simply earns interest at the deterministic rate of 4.5%.

- At the highest ages shown in Figures 5.25-5.27, the graphs are close to exponential curves representing interest being earned at the deterministic rate of 4.5%, as there are few annuity payments still being made at these ages. Thus positive fund sizes become more positive exponentially and negative fund sizes, representing insolvency, become more negative exponentially.

- The black solid curve in each graph, representing the median, has a turning point when the rate at which annuity payments are being made equals the rate at which interest is being earned on the fund. As a result of the improvement of mortality over time, the turning point occurs at higher ages for lower values of the age in 2005.

Table 5.20 calculates the diversification benefit for a portfolio with $N(x) = 1$ for x = 35, 45, 55, 65 and 75, and $N(y) = 0$ for all other ages y, by comparing the ICA capital for this portfolio with the sum of the ICA capital results for the individual ages in Table 5.18. We find that the diversification benefit is greatest for CMI data and least for England and Wales male data. Some understanding of this result can be gained by observing that for CMI data, Table 5.19 shows that the main risk factor at the younger ages, $Z_{2006}^{(3)}$, is different from the main risk factor at the older ages, $Z_{2006}^{(4)}$, whereas for England and Wales male data, Table 5.19 shows that $Z_{2006}^{(3)}$ is the main risk factor at all the ages. In addition, Tables 4.6-4.8 show that the correlations between the different risk factors are lowest for CMI data.

A greater understanding of the sources of the diversification benefits shown in Table 5.20 can be gained from Figures 5.28-5.30 and Tables 5.21-5.23. Figures 5.28, 5.29 and 5.30 are scatter diagrams showing the values of $C(x)$ and $C(y)$ for each pair of ages (x,y) in 2005, under the same 500 sets of values of the $Z_{2006}^{(l)}$ that were used in calculating the annuity values in Section 5.5, for CMI data, England and Wales male data and England and Wales female data respectively. Tables 5.21, 5.22 and 5.23 show the corresponding correlations between $C(x)$ and $C(y)$, for CMI data, England and Wales male data and England and Wales female data respectively. The correlations in Tables 5.21-5.23 were calculated analytically rather than being based on the data points in Figures 5.28-5.30.

The correlations in Tables 5.21-5.23 tend to be higher, and the points in Figures 5.28-5.30 tend to lie closer to a straight line of positive slope, when *x* and *y* are close together than when they are far apart. The correlations also tend to be higher and the points closer to a straight line of positive slope for England and Wales male data compared with England and Wales female data, and for England and Wales female data compared with CMI data. This suggests that pairs of ages *x* and *y* that are close together give rise to relatively little diversification benefit, and is consistent with the observation from Table 5.20 that the overall diversification benefit is greatest for CMI data and least for England and Wales male data. However, the relative magnitudes of *C*(*x*) and *C*(*y*) for different pairs of ages (*x*,*y*) also influence where the diversification benefits come from. As the annuity values increase with age in 2005, with the exception of a decrease from age 65 to 75, a large value of *x* and a large value of *y* may give rise to a comparable diversification benefit to a small value of *x* and a large value of *y*, even though the correlation is lower in the latter case.

Tables 5.24-5.29 and Figures 5.31-5.36 present some investigations into the accuracy of the linear approximations we have used, considering only one age in 2005 at a time. Tables 5.24-5.26 and Figures 5.31-5.33 relate to Approximation 1, while Tables 5.27-5.29 and Figures 5.34-5.36 relate to Approximation 2. A comparable investigation was carried out in relation to Approximation 3, but the results are not shown here as the errors were found to be considerably smaller than in Approximations 1 and 2.

Each row of Tables 5.24-5.29 represents a particular set of values of the $Z_{2006}^{(l)}$. The first ten rows represent randomly generated sets of values. Rows 11, 12, 13 and 14 represent, respectively, the effect of setting $Z_{2006}^{(0)}$, $Z_{2006}^{(3)}$, $Z_{2006}^{(4)}$ and $Z_{2006}^{(5)}$ to 0.1 and the other $Z_{2006}^{(l)}$ to zero. Thus the main purpose of Rows 11-14 is to check that the coefficients in Table 5.19 have been calculated correctly, as with these relatively small values of the $Z_{2006}^{(l)}$, it is unlikely that terms which are second or higher order in the $Z_{2006}^{(l)}$ will be significant. Rows 1-10 then indicate the errors introduced by ignoring terms that are second or higher order in the $Z_{2006}^{(l)}$, for typical sets of values of the $Z_{2006}^{(l)}$ that arise in practice. We should therefore regard a rather smaller margin of error as acceptable in Rows 11-14 than in Rows 1-10.

Column (1) of Tables 5.24-5.29 shows the value of the quantity we are approximating to on the basis that the $Z_{2006}^{(l)}$ are all equal to zero. This is the 'deterministic' probability of surviving 2006, $\overline{p}_{x,2006}$, in Tables 5.24-5.26 and the 50-scenario estimate of the 'deterministic' mean annuity value at the end of 2006, $E[FutAnnVal(x,2006)]$, in Tables 5.27-5.29. In Tables 5.24-5.26, Column (2) shows the linear approximations to the conditional probability of surviving 2006 given the values of the $Z_{2006}^{(l)}$ that are used in the relevant row, while in Tables 5.27-5.29, this column shows the linear approximations to the conditional mean annuity value at the end of 2006 given the values of the $Z_{2006}^{(l)}$ that are used in the relevant row. Column (3) shows the actual probability of surviving 2006 or the actual 50-scenario estimate of the mean annuity value at the end of 2006, as appropriate. Column (4) shows the differences between Columns (2) and (1), while Column (5) shows the differences between Columns (3) and (1). We should like the differences between Columns (2) and (3), or equivalently the differences between Columns (4) and (5), to be small. The size of these differences can be measured either as the absolute error in the approximation of Column (3) by Column (2), which is shown in Column (6), or as the percentage error, which is shown in Column (7).

Figures 5.31-5.36 provide a visual representation of the comparisons between Columns (4) and (5) of Tables 5.24-5.29, for Rows 1-10 only. We should like all the points in these figures to lie close to a straight line with zero intercept and unit slope, which is also shown in the figures.

Rows 11-14 in Tables 5.24-5.29 do not give any reason to doubt that the coefficients of the $Z_{2006}^{(l)}$ have been calculated correctly – there is only one case where the percentage difference between Columns (4) and (5) is greater than 0.5%.

The points in Figures 5.34-5.36 all lie very close to the line with zero intercept and unit slope, providing evidence that the use of Approximation 2 is appropriate. In addition, all but two of the differences between Columns (4) and (5) of Tables 5.27-5.29, as reported in Column (6), are less than 0.001 in absolute value, with most of the differences being significantly less than this. These differences of less than 0.001 should have a probability of surviving 2006 applied to them and should then be discounted for one year to calculate the impact on ICA capital, so the impact on ICA capital is also less than 0.001 in absolute value.

Some of the graphs in Figures 5.31-5.33 reveal outliers. However, the outliers occur at low values of the age in 2005, for which the probability of death in 2006 is small, so it is not clear that there will be a significant impact on ICA capital. In fact, there are only two cases where the difference between Columns (4) and (5) of Tables 5.24-5.26 is greater than 4.5×10^{-5}, and these two cases both occur at age 75 rather than at the younger ages. To calculate the impact on ICA capital, the difference between Columns (4) and (5) needs to be multiplied by an annuity factor, which cannot exceed the value at 4.5% interest of a perpetuity of 1 *p.a.* payable annually in advance, and then discounted for one year. Thus, except in the two cases where the difference is greater than 4.5×10^{-5}, the absolute impact on ICA capital cannot exceed 4.5×10^{-5} multiplied by the value at 4.5% interest of a perpetuity of 1 *p.a.* payable annually in arrears, or $4.5 \times 10^{-5} / 0.045 = 0.001$.

5.7 Conclusion

Prediction intervals have been constructed for the future values of the different time series of parameters for all three data sets. However, the uncertainty in the future mortality projected by our model is not in fact as great as the width of these prediction intervals might suggest, as a result of the correlations between the different kappa parameter series and because a typical scenario will have high parameter values in some future years but low parameter values in others.

To use the mortality projections given by our model to calculate annuity values, it was necessary to consider both extrapolation of the 2005 mortality curve to ages above the highest age in the data and future mortality improvements at ages above the highest age in the data. Two possible assumptions were considered for each of these to provide an indication of the range of possible outcomes.

Values of both immediate and deferred annuities on lives of different ages were presented at different percentiles, at a deterministic interest rate of 4.5%. The means and standard deviations of the distributions of annuity values were also presented. These annuity values were then compared with previously published projections, starting from the 2005 mortality curves fitted in this thesis. It was found that our model generally produced higher annuity values than the previously published projections for CMI data, but lower annuity values for England and Wales data for both males and females. For ages in 2005 below 65, the annuity values from our model increased relative to those from the previously published projections. Plots of projected forces of mortality and probabilities of survival were produced to give an understanding of the sources of these differences.

Risk-based capital calculations under the ICA regime were presented. It was found that our model produced significantly lower capital requirements than a particular 'rule of thumb' that is widely used within the UK life insurance industry. This might suggest that a company currently using the 'rule of thumb' can reasonably justify reducing the amount of ICA capital it holds in respect of longevity risk, but some issues for the company to consider before reducing this capital were highlighted. ICA capital calculations were carried out for portfolios of immediate and deferred annuities at different ages as well as for single ages, and it was found that the diversification benefit to ICA capital from holding a portfolio of annuities at different ages was greatest for CMI data and least for England and Wales male data. Some linear approximations were used in the calculation of the ICA capital amounts, and some test results

were presented to confirm that these approximations were sufficiently accurate for the purpose in hand.

Table 5.1 – Key to Tables 5.2-5.14

Table	Data set	Improvement assumption	Extrapolation method	Number of scenarios
5.2	CMI	Low	Log-linear	500
5.3	CMI	High	Log-linear	500
5.4	E&W Male	Low	Log-linear	500
5.5	E&W Male	High	Log-linear	500
5.6	E&W Female	Low	Log-linear	500
5.7	E&W Female	High	Log-linear	500
5.8	E&W Male	High	Log-linear	5,000
5.9	CMI	Low	LifeMetrics	500
5.10	CMI	High	LifeMetrics	500
5.11	E&W Male	Low	LifeMetrics	500
5.12	E&W Male	High	LifeMetrics	500
5.13	E&W Female	Low	LifeMetrics	500
5.14	E&W Female	High	LifeMetrics	500

Table 5.2 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *CMI data* under the *low improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**4.065**	**6.092**	**9.073**	**14.160**	**10.334**
St. dev.	0.090	0.142	0.211	0.290	0.335
St. dev. / Mean	2.2%	2.3%	2.3%	2.0%	3.2%
0.5th percentile	3.847	5.750	8.487	13.401	9.453
1st percentile	3.871	5.764	8.558	13.470	9.498
2.5th percentile	3.895	5.811	8.625	13.509	9.659
5th percentile	3.917	5.844	8.710	13.626	9.778
10th percentile	3.943	5.893	8.814	13.798	9.887
20th percentile	3.987	5.977	8.905	13.918	10.071
30th percentile	4.019	6.027	8.961	14.014	10.159
40th percentile	4.045	6.061	9.016	14.107	10.262
50th percentile	4.067	6.092	9.072	14.181	10.347
60th percentile	4.087	6.132	9.132	14.248	10.428
70th percentile	4.112	6.169	9.196	14.323	10.508
80th percentile	4.145	6.220	9.253	14.401	10.624
90th percentile	4.178	6.276	9.346	14.520	10.763
95th percentile	**4.208**	**6.317**	**9.407**	**14.611**	**10.871**
97.5th percentile	4.236	6.352	9.457	14.678	10.962
99th percentile	4.265	6.391	9.529	14.778	11.068
99.5th percentile	4.278	6.421	9.563	14.830	11.131

Table 5.3 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *CMI data* under the *high improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**4.209**	**6.249**	**9.224**	**14.305**	**10.458**
St. dev.	0.126	0.183	0.255	0.341	0.389
St. dev. / Mean	3.0%	2.9%	2.8%	2.4%	3.7%
0.5th percentile	3.901	5.801	8.566	13.457	9.499
1st percentile	3.945	5.846	8.652	13.490	9.538
2.5th percentile	3.972	5.884	8.683	13.590	9.698
5th percentile	4.014	5.946	8.824	13.715	9.821
10th percentile	4.046	6.011	8.913	13.868	9.945
20th percentile	4.094	6.095	9.016	14.020	10.139
30th percentile	4.140	6.159	9.093	14.139	10.245
40th percentile	4.177	6.201	9.155	14.234	10.360
50th percentile	4.205	6.247	9.216	14.328	10.471
60th percentile	4.235	6.292	9.290	14.414	10.560
70th percentile	4.273	6.343	9.359	14.493	10.645
80th percentile	4.317	6.403	9.445	14.579	10.791
90th percentile	4.371	6.491	9.545	14.727	10.940
95th percentile	**4.431**	**6.542**	**9.652**	**14.845**	**11.109**
97.5th percentile	4.471	6.597	9.701	14.973	11.239
99th percentile	4.492	6.681	9.794	15.061	11.320
99.5th percentile	4.514	6.715	9.843	15.168	11.413

Table 5.4 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W male data* under the *low improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.329**	**5.008**	**7.624**	**12.546**	**8.803**
St. dev.	0.106	0.144	0.186	0.216	0.214
St. dev. / Mean	3.2%	2.9%	2.4%	1.7%	2.4%
0.5th percentile	3.071	4.664	7.186	12.002	8.269
1st percentile	3.091	4.684	7.225	12.059	8.285
2.5th percentile	3.127	4.716	7.275	12.126	8.392
5th percentile	3.155	4.778	7.317	12.185	8.446
10th percentile	3.190	4.823	7.388	12.282	8.540
20th percentile	3.238	4.891	7.466	12.349	8.633
30th percentile	3.269	4.935	7.518	12.427	8.692
40th percentile	3.296	4.969	7.568	12.493	8.742
50th percentile	3.332	4.993	7.621	12.545	8.793
60th percentile	3.357	5.037	7.670	12.597	8.847
70th percentile	3.384	5.090	7.730	12.651	8.911
80th percentile	3.418	5.132	7.778	12.731	8.991
90th percentile	3.464	5.187	7.860	12.826	9.077
95th percentile	**3.514**	**5.247**	**7.937**	**12.882**	**9.140**
97.5th percentile	3.542	5.296	7.987	12.975	9.240
99th percentile	3.564	5.348	8.045	13.037	9.328
99.5th percentile	3.571	5.360	8.102	13.102	9.394

Table 5.5 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W male data* under the *high improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.374**	**5.055**	**7.669**	**12.591**	**8.842**
St. dev.	0.122	0.161	0.202	0.234	0.233
St. dev. / Mean	3.6%	3.2%	2.6%	1.9%	2.6%
0.5th percentile	3.083	4.680	7.199	12.022	8.275
1st percentile	3.104	4.705	7.234	12.070	8.301
2.5th percentile	3.149	4.742	7.295	12.148	8.406
5th percentile	3.179	4.793	7.345	12.216	8.474
10th percentile	3.218	4.857	7.414	12.311	8.553
20th percentile	3.268	4.925	7.493	12.374	8.659
30th percentile	3.302	4.970	7.554	12.471	8.722
40th percentile	3.340	5.008	7.604	12.532	8.768
50th percentile	3.377	5.044	7.659	12.592	8.827
60th percentile	3.404	5.089	7.716	12.646	8.890
70th percentile	3.435	5.142	7.778	12.699	8.955
80th percentile	3.474	5.190	7.840	12.783	9.035
90th percentile	3.527	5.263	7.924	12.904	9.142
95th percentile	**3.595**	**5.332**	**8.016**	**12.956**	**9.219**
97.5th percentile	3.627	5.391	8.068	13.069	9.325
99th percentile	3.660	5.441	8.150	13.188	9.430
99.5th percentile	3.678	5.468	8.200	13.219	9.494

Table 5.6 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W female data* under the *low improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.539**	**5.442**	**8.348**	**13.620**	**9.772**
St. dev.	0.115	0.153	0.198	0.236	0.239
St. dev. / Mean	3.3%	2.8%	2.4%	1.7%	2.4%
0.5th percentile	3.249	5.068	7.845	13.030	9.178
1st percentile	3.273	5.075	7.886	13.077	9.210
2.5th percentile	3.316	5.143	7.952	13.112	9.287
5th percentile	3.349	5.192	8.014	13.239	9.370
10th percentile	3.393	5.247	8.110	13.327	9.452
20th percentile	3.442	5.321	8.183	13.429	9.578
30th percentile	3.475	5.362	8.241	13.490	9.648
40th percentile	3.509	5.399	8.286	13.561	9.709
50th percentile	3.541	5.436	8.337	13.628	9.777
60th percentile	3.571	5.474	8.396	13.695	9.838
70th percentile	3.599	5.527	8.458	13.743	9.897
80th percentile	3.636	5.574	8.520	13.820	9.982
90th percentile	3.690	5.639	8.596	13.916	10.079
95th percentile	**3.736**	**5.691**	**8.671**	**14.014**	**10.145**
97.5th percentile	3.762	5.748	8.744	14.057	10.232
99th percentile	3.784	5.787	8.796	14.133	10.320
99.5th percentile	3.811	5.818	8.846	14.175	10.343

Table 5.7 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W female data* under the *high improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.566**	**5.473**	**8.379**	**13.656**	**9.805**
St. dev.	0.128	0.167	0.213	0.253	0.257
St. dev. / Mean	3.6%	3.1%	2.5%	1.9%	2.6%
0.5th percentile	3.253	5.074	7.853	13.033	9.179
1st percentile	3.278	5.086	7.898	13.090	9.205
2.5th percentile	3.326	5.156	7.963	13.127	9.292
5th percentile	3.359	5.207	8.037	13.248	9.382
10th percentile	3.408	5.268	8.128	13.349	9.466
20th percentile	3.457	5.339	8.207	13.446	9.598
30th percentile	3.494	5.383	8.265	13.516	9.665
40th percentile	3.532	5.426	8.313	13.593	9.732
50th percentile	3.569	5.464	8.360	13.657	9.809
60th percentile	3.599	5.508	8.429	13.730	9.877
70th percentile	3.628	5.559	8.497	13.782	9.931
80th percentile	3.670	5.616	8.571	13.870	10.022
90th percentile	3.732	5.685	8.644	13.979	10.142
95th percentile	**3.783**	**5.749**	**8.722**	**14.089**	**10.212**
97.5th percentile	3.827	5.807	8.815	14.139	10.305
99th percentile	3.845	5.855	8.890	14.217	10.408
99.5th percentile	3.874	5.892	8.919	14.305	10.440

Table 5.8 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W male data* under the *high improvement assumption* and the *log-linear extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *5,000 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.371**	**5.051**	**7.662**	**12.580**	**8.829**
St. dev.	0.122	0.162	0.202	0.230	0.228
St. dev. / Mean	3.6%	3.2%	2.6%	1.8%	2.6%
0.5th percentile	3.051	4.642	7.138	11.988	8.263
1st percentile	3.083	4.666	7.194	12.063	8.307
2.5th percentile	3.129	4.729	7.265	12.147	8.401
5th percentile	3.167	4.778	7.334	12.211	8.462
10th percentile	3.211	4.841	7.406	12.289	8.542
20th percentile	3.266	4.916	7.490	12.385	8.636
30th percentile	3.309	4.968	7.556	12.456	8.707
40th percentile	3.342	5.011	7.607	12.522	8.765
50th percentile	3.373	5.049	7.660	12.576	8.821
60th percentile	3.402	5.093	7.715	12.638	8.880
70th percentile	3.437	5.138	7.768	12.698	8.945
80th percentile	3.475	5.187	7.833	12.772	9.020
90th percentile	3.527	5.260	7.923	12.880	9.127
95th percentile	**3.568**	**5.317**	**7.991**	**12.957**	**9.208**
97.5th percentile	3.606	5.362	8.053	13.035	9.291
99th percentile	3.650	5.426	8.131	13.132	9.407
99.5th percentile	3.677	5.466	8.172	13.192	9.449

Table 5.9 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *CMI data* under the *low improvement assumption* and the *LifeMetrics extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**4.058**	**6.082**	**9.059**	**14.140**	**10.306**
St. dev.	0.089	0.141	0.210	0.288	0.331
St. dev. / Mean	2.2%	2.3%	2.3%	2.0%	3.2%
0.5th percentile	3.841	5.741	8.477	13.387	9.433
1st percentile	3.865	5.756	8.549	13.458	9.480
2.5th percentile	3.889	5.802	8.613	13.493	9.638
5th percentile	3.911	5.834	8.697	13.610	9.754
10th percentile	3.936	5.884	8.803	13.781	9.863
20th percentile	3.980	5.967	8.891	13.900	10.046
30th percentile	4.011	6.017	8.948	13.994	10.132
40th percentile	4.038	6.051	9.003	14.088	10.234
50th percentile	4.060	6.081	9.058	14.160	10.319
60th percentile	4.079	6.121	9.118	14.228	10.399
70th percentile	4.105	6.158	9.180	14.302	10.478
80th percentile	4.138	6.209	9.238	14.380	10.592
90th percentile	4.171	6.265	9.330	14.497	10.731
95th percentile	**4.200**	**6.306**	**9.391**	**14.588**	**10.838**
97.5th percentile	4.228	6.340	9.440	14.655	10.927
99th percentile	4.257	6.379	9.513	14.754	11.033
99.5th percentile	4.270	6.409	9.546	14.806	11.095

Table 5.10 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *CMI data* under the *high improvement assumption* and the *LifeMetrics extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**4.209**	**6.247**	**9.216**	**14.290**	**10.432**
St. dev.	0.128	0.185	0.257	0.342	0.388
St. dev. / Mean	3.0%	3.0%	2.8%	2.4%	3.7%
0.5th percentile	3.896	5.793	8.556	13.443	9.479
1st percentile	3.943	5.839	8.645	13.475	9.517
2.5th percentile	3.970	5.877	8.671	13.575	9.676
5th percentile	4.010	5.941	8.814	13.700	9.797
10th percentile	4.044	6.006	8.903	13.852	9.921
20th percentile	4.093	6.090	9.009	14.006	10.113
30th percentile	4.140	6.155	9.085	14.122	10.220
40th percentile	4.176	6.199	9.147	14.219	10.332
50th percentile	4.205	6.244	9.210	14.311	10.444
60th percentile	4.236	6.290	9.281	14.400	10.535
70th percentile	4.274	6.342	9.352	14.481	10.618
80th percentile	4.319	6.401	9.440	14.566	10.762
90th percentile	4.372	6.491	9.540	14.713	10.914
95th percentile	**4.433**	**6.550**	**9.648**	**14.830**	**11.082**
97.5th percentile	4.468	6.600	9.700	14.965	11.220
99th percentile	4.496	6.686	9.787	15.060	11.295
99.5th percentile	4.516	6.720	9.848	15.159	11.394

Table 5.11 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W male data* under the *low improvement assumption* and the *LifeMetrics extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.334**	**5.015**	**7.632**	**12.556**	**8.815**
St. dev.	0.107	0.146	0.187	0.219	0.217
St. dev. / Mean	3.2%	2.9%	2.5%	1.7%	2.5%
0.5th percentile	3.073	4.667	7.190	12.008	8.276
1st percentile	3.094	4.688	7.228	12.065	8.292
2.5th percentile	3.130	4.720	7.280	12.133	8.400
5th percentile	3.158	4.781	7.323	12.192	8.454
10th percentile	3.194	4.827	7.393	12.289	8.550
20th percentile	3.243	4.899	7.472	12.357	8.642
30th percentile	3.273	4.940	7.525	12.438	8.703
40th percentile	3.302	4.975	7.575	12.503	8.753
50th percentile	3.338	5.000	7.628	12.556	8.805
60th percentile	3.363	5.044	7.680	12.608	8.861
70th percentile	3.390	5.097	7.738	12.662	8.927
80th percentile	3.424	5.140	7.788	12.743	9.006
90th percentile	3.472	5.195	7.870	12.839	9.091
95th percentile	**3.521**	**5.259**	**7.949**	**12.895**	**9.156**
97.5th percentile	3.550	5.305	7.997	12.990	9.259
99th percentile	3.573	5.359	8.057	13.058	9.350
99.5th percentile	3.580	5.372	8.114	13.119	9.415

Table 5.12 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W male data* under the *high improvement assumption* and the *LifeMetrics extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.389**	**5.071**	**7.684**	**12.609**	**8.860**
St. dev.	0.127	0.166	0.207	0.240	0.239
St. dev. / Mean	3.7%	3.3%	2.7%	1.9%	2.7%
0.5th percentile	3.087	4.687	7.204	12.031	8.281
1st percentile	3.109	4.711	7.239	12.077	8.312
2.5th percentile	3.159	4.750	7.303	12.157	8.415
5th percentile	3.190	4.801	7.354	12.224	8.482
10th percentile	3.229	4.866	7.425	12.319	8.563
20th percentile	3.279	4.935	7.505	12.391	8.674
30th percentile	3.315	4.978	7.562	12.484	8.737
40th percentile	3.352	5.022	7.621	12.546	8.786
50th percentile	3.391	5.058	7.674	12.610	8.843
60th percentile	3.418	5.105	7.733	12.663	8.910
70th percentile	3.452	5.159	7.797	12.717	8.976
80th percentile	3.492	5.206	7.861	12.800	9.057
90th percentile	3.549	5.286	7.942	12.929	9.168
95th percentile	**3.620**	**5.360**	**8.036**	**12.987**	**9.250**
97.5th percentile	3.651	5.419	8.095	13.101	9.358
99th percentile	3.686	5.469	8.184	13.229	9.466
99.5th percentile	3.705	5.500	8.232	13.268	9.531

Table 5.13 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W female data* under the *low improvement assumption* and the *LifeMetrics extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.547**	**5.452**	**8.360**	**13.638**	**9.794**
St. dev.	0.117	0.155	0.201	0.240	0.244
St. dev. / Mean	3.3%	2.9%	2.4%	1.8%	2.5%
0.5th percentile	3.252	5.073	7.852	13.038	9.189
1st percentile	3.277	5.079	7.894	13.089	9.220
2.5th percentile	3.321	5.150	7.959	13.125	9.300
5th percentile	3.354	5.199	8.024	13.250	9.386
10th percentile	3.399	5.254	8.118	13.341	9.469
20th percentile	3.448	5.330	8.194	13.444	9.594
30th percentile	3.482	5.370	8.251	13.507	9.665
40th percentile	3.516	5.409	8.298	13.578	9.730
50th percentile	3.549	5.445	8.348	13.644	9.797
60th percentile	3.579	5.485	8.408	13.714	9.864
70th percentile	3.607	5.538	8.472	13.763	9.920
80th percentile	3.645	5.586	8.534	13.844	10.008
90th percentile	3.700	5.653	8.613	13.941	10.111
95th percentile	**3.748**	**5.706**	**8.687**	**14.043**	**10.178**
97.5th percentile	3.774	5.763	8.762	14.085	10.264
99th percentile	3.795	5.803	8.818	14.162	10.355
99.5th percentile	3.823	5.834	8.866	14.205	10.380

Table 5.14 – Means, standard deviations and key percentiles of the empirical distributions of deferred/immediate annuity values for *E&W female data* under the *high improvement assumption* and the *LifeMetrics extrapolation method* – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest – *500 scenarios*

Quantity	Age 35 in 2005 (Deferred)	Age 45 in 2005 (Deferred)	Age 55 in 2005 (Deferred)	Age 65 in 2005 (Immediate)	Age 75 in 2005 (Immediate)
Mean	**3.580**	**5.490**	**8.398**	**13.682**	**9.834**
St. dev.	0.133	0.173	0.220	0.261	0.265
St. dev. / Mean	3.7%	3.2%	2.6%	1.9%	2.7%
0.5th percentile	3.257	5.078	7.861	13.042	9.191
1st percentile	3.282	5.096	7.908	13.102	9.215
2.5th percentile	3.333	5.164	7.970	13.143	9.306
5th percentile	3.366	5.216	8.051	13.259	9.397
10th percentile	3.416	5.279	8.137	13.369	9.486
20th percentile	3.466	5.350	8.218	13.463	9.623
30th percentile	3.504	5.395	8.279	13.539	9.689
40th percentile	3.543	5.441	8.327	13.617	9.758
50th percentile	3.583	5.480	8.378	13.683	9.838
60th percentile	3.612	5.526	8.447	13.753	9.905
70th percentile	3.643	5.579	8.523	13.812	9.962
80th percentile	3.688	5.636	8.592	13.904	10.053
90th percentile	3.753	5.710	8.671	14.012	10.178
95th percentile	**3.810**	**5.783**	**8.750**	**14.130**	**10.255**
97.5th percentile	3.855	5.844	8.843	14.185	10.357
99th percentile	3.875	5.888	8.929	14.263	10.466
99.5th percentile	3.902	5.926	8.955	14.375	10.493

Table 5.15 – Deferred/immediate annuity values based on previously published projections applied to the 2005 CMI mortality curve fitted in this thesis, using the log-linear extrapolation method – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest

Age in 2005	Medium Cohort	Medium Cohort subject to 1% underpin	Long Cohort
35 (Deferred)	3.717	3.820	3.836
45 (Deferred)	5.673	5.780	5.857
55 (Deferred)	8.608	8.706	8.892
65 (Immediate)	13.723	13.808	14.182
75 (Immediate)	10.130	10.189	10.642

Table 5.16 – Deferred/immediate annuity values based on GAD projections applied to the 2005 England and Wales male mortality curve fitted in this thesis, using the log-linear extrapolation method – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest

Age in 2005	LLE	Principal	HLE
35 (Deferred)	2.984	3.298	3.625
45 (Deferred)	4.705	5.034	5.384
55 (Deferred)	7.515	7.812	8.132
65 (Immediate)	12.633	12.850	13.085
75 (Immediate)	8.817	8.930	9.050

Table 5.17 – Deferred/immediate annuity values based on GAD projections applied to the 2005 England and Wales female mortality curve fitted in this thesis, using the log-linear extrapolation method – level single life annuity of 1 *p.a.* payable annually in advance from age 65 and valued using 4.5% *p.a.* interest

Age in 2005	LLE	Principal	HLE
35 (Deferred)	3.429	3.672	3.924
45 (Deferred)	5.393	5.641	5.904
55 (Deferred)	8.444	8.661	8.893
65 (Immediate)	13.873	14.026	14.191
75 (Immediate)	9.820	9.894	9.971

Table 5.18 – ICA capital calculations for deferred/immediate annuities and comparisons with the annuity value distributions obtained in Section 5.5, as shown in Table 5.3 for CMI data, Table 5.5 for England and Wales male data and Table 5.7 for England and Wales female data

Data set	Age in 2005	Mean annuity value from Section 5.5	ICA capital	Mean annuity value + ICA capital	Percentile of distribution in Section 5.5	95th percentile
CMI	35	4.209	0.047	4.256	64%	4.431
E&W Male	35	3.374	0.055	3.430	69%	3.595
E&W Female	35	3.566	0.057	3.623	69%	3.783
CMI	45	6.249	0.075	6.323	67%	6.542
E&W Male	45	5.055	0.084	5.140	69%	5.332
E&W Female	45	5.473	0.083	5.557	69%	5.749
CMI	55	9.224	0.125	9.349	69%	9.652
E&W Male	55	7.669	0.128	7.797	73%	8.016
E&W Female	55	8.379	0.126	8.505	71%	8.722
CMI	65	14.305	0.218	14.523	73%	14.845
E&W Male	65	12.591	0.184	12.775	80%	12.956
E&W Female	65	13.656	0.185	13.841	76%	14.089
CMI	75	10.458	0.336	10.794	80%	11.109
E&W Male	75	8.842	0.234	9.076	85%	9.219
E&W Female	75	9.805	0.246	10.051	83%	10.212

Table 5.19 – Coefficients of the $Z_{2006}^{(i)}$, for i = 0,3,4,5, in the expressions for $C(x)$ for each age x and each data set

Data set	Age in 2005, x	Coefficient of $Z_{2006}^{(0)}$	Coefficient of $Z_{2006}^{(3)}$	Coefficient of $Z_{2006}^{(4)}$	Coefficient of $Z_{2006}^{(5)}$
CMI	35	-0.00127	-0.02316	-0.00689	0.00097
E&W Male	35	-0.00280	-0.02603	-0.00282	0.00213
E&W Female	35	-0.00490	-0.02883	-0.00427	0.00136
CMI	45	-0.00189	-0.03861	-0.01728	0.00377
E&W Male	45	-0.00381	-0.04098	-0.00768	0.00589
E&W Female	45	-0.00664	-0.04391	-0.01196	0.00373
CMI	55	-0.00285	-0.06245	-0.04386	0.00924
E&W Male	55	-0.00478	-0.06209	-0.02050	0.01149
E&W Female	55	-0.00849	-0.06567	-0.03047	0.00632
CMI	65	-0.00331	-0.08950	-0.10241	-0.00180
E&W Male	65	-0.00422	-0.08154	-0.04728	0.00479
E&W Female	65	-0.00744	-0.08528	-0.06634	-0.00197
CMI	75	-0.00219	-0.10451	-0.18576	-0.06768
E&W Male	75	-0.00233	-0.08440	-0.07882	-0.03246
E&W Female	75	-0.00433	-0.09226	-0.11122	-0.03176

Table 5.20 – Diversification benefits from a portfolio of deferred/immediate annuities with equal numbers of annuitants aged 35, 45, 55, 65 and 75 in 2005, compared with each age separately

Data set	Undiversified ICA capital	Diversified ICA capital	Diversification benefit
CMI	0.8016	0.7526	0.0490
E&W Male	0.6852	0.6670	0.0182
E&W Female	0.6975	0.6746	0.0229

Table 5.21 – Correlation coefficients between $C(x)$ and $C(y)$ for different ages x and y, for CMI data

x/y	35	45	55	65	75
35	1				
45	0.8056	1			
55	0.4206	0.6667	1		
65	0.0477	0.2615	0.6021	1	
75	-0.0482	0.0421	0.2237	0.5722	1

Table 5.22 – Correlation coefficients between $C(x)$ and $C(y)$ for different ages x and y, for England and Wales male data

x/y	35	45	55	65	75
35	1				
45	0.9012	1			
55	0.7474	0.8620	1		
65	0.6395	0.7398	0.8734	1	
75	0.6867	0.7296	0.7905	0.8953	1

Table 5.23 – Correlation coefficients between $C(x)$ and $C(y)$ for different ages x and y, for England and Wales female data

x/y	35	45	55	65	75
35	1				
45	0.8557	1			
55	0.6339	0.8016	1		
65	0.4216	0.5751	0.7796	1	
75	0.3505	0.4513	0.5999	0.8025	1

Table 5.24 – Investigation of the accuracy of Approximation 1 for CMI data – age in 2005 (i) 35, (ii) 45, (iii) 55 – column (1) = 'deterministic' probability of surviving 2006, (2) = linear approximation to conditional probability of surviving 2006, (3) = actual conditional probability of surviving 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(i)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9994846	0.9994813	0.9994807	-0.0000033	-0.0000040	0.0000007	0.00007%
2	0.9994846	0.9994822	0.9994806	-0.0000024	-0.0000040	0.0000016	0.00016%
3	0.9994846	0.9995019	0.9995019	0.0000173	0.0000173	0.0000000	0.00000%
4	0.9994846	0.9994488	0.9994484	-0.0000358	-0.0000362	0.0000004	0.00004%
5	0.9994846	0.9995180	0.9995179	0.0000333	0.0000332	0.0000001	0.00001%
6	0.9994846	0.9993800	0.9993797	-0.0001046	-0.0001049	0.0000003	0.00003%
7	0.9994846	0.9994415	0.9994404	-0.0000431	-0.0000442	0.0000011	0.00011%
8	0.9994846	0.9994569	0.9994568	-0.0000278	-0.0000278	0.0000000	0.00000%
9	0.9994846	0.9996093	0.9996009	0.0001247	0.0001162	0.0000085	0.00085%
10	0.9994846	0.9994455	0.9994455	-0.0000391	-0.0000391	0.0000000	0.00000%
11	0.9994846	0.9994770	0.9994770	-0.0000076	-0.0000076	0.0000000	0.00000%
12	0.9994846	0.9994840	0.9994840	-0.0000006	-0.0000006	0.0000000	0.00000%
13	0.9994846	0.9994856	0.9994856	0.0000010	0.0000009	0.0000000	0.00000%
14	0.9994846	0.9994841	0.9994841	-0.0000005	-0.0000005	0.0000000	0.00000%

(ii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9990114	0.9989917	0.9989910	-0.0000197	-0.0000204	0.0000007	0.00007%
2	0.9990114	0.9989708	0.9989679	-0.0000406	-0.0000435	0.0000029	0.00029%
3	0.9990114	0.9990130	0.9990129	0.0000016	0.0000014	0.0000002	0.00002%
4	0.9990114	0.9989603	0.9989597	-0.0000511	-0.0000517	0.0000005	0.00005%
5	0.9990114	0.9990417	0.9990417	0.0000303	0.0000303	0.0000000	0.00000%
6	0.9990114	0.9988982	0.9988980	-0.0001132	-0.0001134	0.0000002	0.00002%
7	0.9990114	0.9989439	0.9989424	-0.0000675	-0.0000690	0.0000015	0.00015%
8	0.9990114	0.9990068	0.9990063	-0.0000046	-0.0000051	0.0000005	0.00005%
9	0.9990114	0.9990779	0.9990686	0.0000665	0.0000572	0.0000094	0.00094%
10	0.9990114	0.9989819	0.9989818	-0.0000295	-0.0000296	0.0000001	0.00001%
11	0.9990114	0.9990038	0.9990038	-0.0000076	-0.0000076	0.0000000	0.00000%
12	0.9990114	0.9990089	0.9990089	-0.0000025	-0.0000025	0.0000000	0.00000%
13	0.9990114	0.9990139	0.9990139	0.0000025	0.0000025	0.0000000	0.00000%
14	0.9990114	0.9990120	0.9990120	0.0000006	0.0000006	0.0000000	0.00000%

(iii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9970823	0.9970386	0.9970381	-0.0000436	-0.0000442	0.0000006	0.00006%
2	0.9970823	0.9969422	0.9969371	-0.0001400	-0.0001451	0.0000051	0.00051%
3	0.9970823	0.9969787	0.9969758	-0.0001036	-0.0001064	0.0000029	0.00029%
4	0.9970823	0.9970164	0.9970161	-0.0000659	-0.0000662	0.0000003	0.00003%
5	0.9970823	0.9970599	0.9970594	-0.0000223	-0.0000229	0.0000005	0.00005%
6	0.9970823	0.9969642	0.9969641	-0.0001181	-0.0001181	0.0000001	0.00001%
7	0.9970823	0.9969905	0.9969896	-0.0000918	-0.0000926	0.0000008	0.00008%
8	0.9970823	0.9971752	0.9971722	0.0000930	0.0000899	0.0000030	0.00030%
9	0.9970823	0.9971156	0.9971114	0.0000333	0.0000292	0.0000041	0.00041%
10	0.9970823	0.9970901	0.9970896	0.0000079	0.0000074	0.0000005	0.00005%
11	0.9970823	0.9970745	0.9970745	-0.0000078	-0.0000078	0.0000000	0.00000%
12	0.9970823	0.9970719	0.9970718	-0.0000104	-0.0000104	0.0000000	0.00000%
13	0.9970823	0.9970857	0.9970857	0.0000035	0.0000034	0.0000000	0.00000%
14	0.9970823	0.9970904	0.9970904	0.0000082	0.0000082	0.0000000	0.00000%

Table 5.24 (continued) **–** Investigation of the accuracy of Approximation 1 for CMI data – age in 2005 (iv) 65, (v) 75 – column (1) = 'deterministic' probability of surviving 2006, (2) = linear approximation to conditional probability of surviving 2006, (3) = actual conditional probability of surviving 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(iv)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9908489	0.9907333	0.9907324	-0.0001156	-0.0001165	0.0000009	0.00009%
2	0.9908489	0.9904451	0.9904348	-0.0004039	-0.0004141	0.0000103	0.00104%
3	0.9908489	0.9903276	0.9903109	-0.0005213	-0.0005380	0.0000167	0.00168%
4	0.9908489	0.9908671	0.9908670	0.0000182	0.0000181	0.0000001	0.00001%
5	0.9908489	0.9905800	0.9905750	-0.0002689	-0.0002740	0.0000051	0.00051%
6	0.9908489	0.9907531	0.9907531	-0.0000958	-0.0000958	0.0000000	0.00000%
7	0.9908489	0.9908468	0.9908468	-0.0000021	-0.0000022	0.0000000	0.00000%
8	0.9908489	0.9911812	0.9911738	0.0003322	0.0003248	0.0000074	0.00074%
9	0.9908489	0.9911045	0.9911041	0.0002555	0.0002551	0.0000004	0.00004%
10	0.9908489	0.9909558	0.9909546	0.0001069	0.0001057	0.0000012	0.00012%
11	0.9908489	0.9908415	0.9908415	-0.0000075	-0.0000075	0.0000000	0.00000%
12	0.9908489	0.9908130	0.9908129	-0.0000360	-0.0000361	0.0000001	0.00001%
13	0.9908489	0.9908370	0.9908370	-0.0000119	-0.0000120	0.0000000	0.00000%
14	0.9908489	0.9908772	0.9908772	0.0000283	0.0000282	0.0000000	0.00000%

(v)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9713826	0.9706270	0.9706167	-0.0007555	-0.0007658	0.0000103	0.00106%
2	0.9713826	0.9699484	0.9699110	-0.0014342	-0.0014716	0.0000374	0.00385%
3	0.9713826	0.9695911	0.9695330	-0.0017915	-0.0018496	0.0000581	0.00599%
4	0.9713826	0.9719137	0.9719083	0.0005311	0.0005257	0.0000054	0.00055%
5	0.9713826	0.9703751	0.9703562	-0.0010075	-0.0010264	0.0000189	0.00195%
6	0.9713826	0.9711205	0.9711200	-0.0002621	-0.0002625	0.0000005	0.00005%
7	0.9713826	0.9718347	0.9718308	0.0004522	0.0004482	0.0000040	0.00041%
8	0.9713826	0.9718817	0.9718769	0.0004992	0.0004943	0.0000049	0.00050%
9	0.9713826	0.9723751	0.9723634	0.0009925	0.0009808	0.0000117	0.00120%
10	0.9713826	0.9716112	0.9716099	0.0002286	0.0002274	0.0000013	0.00013%
11	0.9713826	0.9713753	0.9713753	-0.0000073	-0.0000073	0.0000000	0.00000%
12	0.9713826	0.9712679	0.9712676	-0.0001147	-0.0001149	0.0000002	0.00002%
13	0.9713826	0.9712684	0.9712681	-0.0001142	-0.0001144	0.0000002	0.00002%
14	0.9713826	0.9714095	0.9714094	0.0000269	0.0000269	0.0000000	0.00000%

Table 5.25 – Investigation of the accuracy of Approximation 1 for England and Wales male data – age in 2005 (i) 35, (ii) 45, (iii) 55 – column (1) = 'deterministic' probability of surviving 2006, (2) = linear approximation to conditional probability of surviving 2006, (3) = actual conditional probability of surviving 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(i)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9989185	0.9989045	0.9989042	-0.0000140	-0.0000144	0.0000003	0.00003%
2	0.9989185	0.9989009	0.9989002	-0.0000176	-0.0000184	0.0000007	0.00007%
3	0.9989185	0.9989362	0.9989361	0.0000176	0.0000176	0.0000000	0.00000%
4	0.9989185	0.9988888	0.9988887	-0.0000297	-0.0000298	0.0000001	0.00001%
5	0.9989185	0.9989448	0.9989448	0.0000263	0.0000262	0.0000000	0.00000%
6	0.9989185	0.9988576	0.9988575	-0.0000610	-0.0000610	0.0000000	0.00000%
7	0.9989185	0.9988778	0.9988774	-0.0000408	-0.0000412	0.0000004	0.00004%
8	0.9989185	0.9989038	0.9989038	-0.0000147	-0.0000148	0.0000000	0.00000%
9	0.9989185	0.9989290	0.9989190	0.0000105	0.0000005	0.0000100	0.00100%
10	0.9989185	0.9988994	0.9988993	-0.0000191	-0.0000192	0.0000001	0.00001%
11	0.9989185	0.9989131	0.9989131	-0.0000054	-0.0000054	0.0000000	0.00000%
12	0.9989185	0.9989167	0.9989167	-0.0000018	-0.0000018	0.0000000	0.00000%
13	0.9989185	0.9989205	0.9989205	0.0000020	0.0000020	0.0000000	0.00000%
14	0.9989185	0.9989173	0.9989173	-0.0000012	-0.0000012	0.0000000	0.00000%

(ii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9977385	0.9977071	0.9977066	-0.0000315	-0.0000319	0.0000004	0.00004%
2	0.9977385	0.9976738	0.9976721	-0.0000648	-0.0000664	0.0000017	0.00017%
3	0.9977385	0.9977301	0.9977300	-0.0000084	-0.0000085	0.0000001	0.00001%
4	0.9977385	0.9976909	0.9976906	-0.0000476	-0.0000479	0.0000002	0.00002%
5	0.9977385	0.9977569	0.9977569	0.0000184	0.0000184	0.0000000	0.00000%
6	0.9977385	0.9976667	0.9976667	-0.0000718	-0.0000718	0.0000000	0.00000%
7	0.9977385	0.9976700	0.9976693	-0.0000685	-0.0000692	0.0000007	0.00007%
8	0.9977385	0.9977574	0.9977569	0.0000188	0.0000184	0.0000004	0.00004%
9	0.9977385	0.9976922	0.9976843	-0.0000463	-0.0000542	0.0000079	0.00079%
10	0.9977385	0.9977321	0.9977320	-0.0000064	-0.0000065	0.0000001	0.00001%
11	0.9977385	0.9977331	0.9977331	-0.0000054	-0.0000054	0.0000000	0.00000%
12	0.9977385	0.9977334	0.9977334	-0.0000051	-0.0000051	0.0000000	0.00000%
13	0.9977385	0.9977417	0.9977417	0.0000032	0.0000032	0.0000000	0.00000%
14	0.9977385	0.9977395	0.9977395	0.0000010	0.0000010	0.0000000	0.00000%

(iii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9943192	0.9942376	0.9942369	-0.0000816	-0.0000824	0.0000008	0.00008%
2	0.9943192	0.9941118	0.9941071	-0.0002074	-0.0002121	0.0000047	0.00047%
3	0.9943192	0.9941936	0.9941918	-0.0001256	-0.0001274	0.0000018	0.00018%
4	0.9943192	0.9942373	0.9942369	-0.0000819	-0.0000823	0.0000004	0.00004%
5	0.9943192	0.9942916	0.9942914	-0.0000276	-0.0000279	0.0000002	0.00002%
6	0.9943192	0.9941949	0.9941946	-0.0001243	-0.0001246	0.0000003	0.00003%
7	0.9943192	0.9941944	0.9941933	-0.0001248	-0.0001259	0.0000011	0.00011%
8	0.9943192	0.9944371	0.9944353	0.0001179	0.0001161	0.0000018	0.00019%
9	0.9943192	0.9942305	0.9942261	-0.0000888	-0.0000931	0.0000044	0.00044%
10	0.9943192	0.9943429	0.9943426	0.0000237	0.0000234	0.0000003	0.00003%
11	0.9943192	0.9943137	0.9943137	-0.0000055	-0.0000055	0.0000000	0.00000%
12	0.9943192	0.9943046	0.9943046	-0.0000146	-0.0000146	0.0000000	0.00000%
13	0.9943192	0.9943221	0.9943221	0.0000028	0.0000028	0.0000000	0.00000%
14	0.9943192	0.9943282	0.9943282	0.0000090	0.0000090	0.0000000	0.00000%

Table 5.25 (continued) **–** Investigation of the accuracy of Approximation 1 for England and Wales male data – age in 2005 (iv) 65, (v) 75 – column (1) = 'deterministic' probability of surviving 2006, (2) = linear approximation to conditional probability of surviving 2006, (3) = actual conditional probability of surviving 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(iv)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9854343	0.9851603	0.9851575	-0.0002739	-0.0002767	0.0000028	0.00028%
2	0.9854343	0.9848010	0.9847861	-0.0006333	-0.0006481	0.0000148	0.00151%
3	0.9854343	0.9849282	0.9849187	-0.0005061	-0.0005156	0.0000095	0.00096%
4	0.9854343	0.9853371	0.9853369	-0.0000971	-0.0000974	0.0000002	0.00002%
5	0.9854343	0.9852346	0.9852329	-0.0001996	-0.0002013	0.0000017	0.00017%
6	0.9854343	0.9851685	0.9851671	-0.0002658	-0.0002672	0.0000014	0.00014%
7	0.9854343	0.9852360	0.9852348	-0.0001983	-0.0001994	0.0000011	0.00012%
8	0.9854343	0.9857621	0.9857579	0.0003279	0.0003236	0.0000042	0.00043%
9	0.9854343	0.9853184	0.9853165	-0.0001158	-0.0001178	0.0000019	0.00020%
10	0.9854343	0.9855286	0.9855281	0.0000944	0.0000938	0.0000005	0.00005%
11	0.9854343	0.9854290	0.9854290	-0.0000052	-0.0000052	0.0000000	0.00000%
12	0.9854343	0.9853952	0.9853951	-0.0000391	-0.0000391	0.0000001	0.00001%
13	0.9854343	0.9854250	0.9854250	-0.0000093	-0.0000093	0.0000000	0.00000%
14	0.9854343	0.9854574	0.9854574	0.0000232	0.0000232	0.0000000	0.00000%

(v)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9592526	0.9580728	0.9580556	-0.0011797	-0.0011970	0.0000173	0.00180%
2	0.9592526	0.9571410	0.9570853	-0.0021116	-0.0021673	0.0000557	0.00582%
3	0.9592526	0.9575823	0.9575475	-0.0016703	-0.0017050	0.0000348	0.00363%
4	0.9592526	0.9593683	0.9593681	0.0001157	0.0001155	0.0000002	0.00002%
5	0.9592526	0.9584759	0.9584681	-0.0007767	-0.0007845	0.0000078	0.00081%
6	0.9592526	0.9586729	0.9586698	-0.0005797	-0.0005828	0.0000032	0.00033%
7	0.9592526	0.9590387	0.9590382	-0.0002139	-0.0002144	0.0000005	0.00005%
8	0.9592526	0.9598799	0.9598748	0.0006273	0.0006223	0.0000051	0.00053%
9	0.9592526	0.9585970	0.9585896	-0.0006556	-0.0006630	0.0000074	0.00077%
10	0.9592526	0.9595428	0.9595416	0.0002902	0.0002890	0.0000012	0.00013%
11	0.9592526	0.9592474	0.9592474	-0.0000052	-0.0000052	0.0000000	0.00000%
12	0.9592526	0.9591427	0.9591426	-0.0001099	-0.0001100	0.0000001	0.00002%
13	0.9592526	0.9591790	0.9591789	-0.0000736	-0.0000737	0.0000001	0.00001%
14	0.9592526	0.9592670	0.9592670	0.0000144	0.0000144	0.0000000	0.00000%

Table 5.26 – Investigation of the accuracy of Approximation 1 for England and Wales female data – age in 2005 (i) 35, (ii) 45, (iii) 55 – column (1) = 'deterministic' probability of surviving 2006, (2) = linear approximation to conditional probability of surviving 2006, (3) = actual conditional probability of surviving 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(i)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9994206	0.9994049	0.9994046	-0.0000157	-0.0000160	0.0000004	0.00004%
2	0.9994206	0.9994011	0.9994004	-0.0000195	-0.0000203	0.0000008	0.00008%
3	0.9994206	0.9994415	0.9994415	0.0000209	0.0000208	0.0000000	0.00000%
4	0.9994206	0.9993899	0.9993899	-0.0000307	-0.0000307	0.0000000	0.00000%
5	0.9994206	0.9994481	0.9994481	0.0000275	0.0000275	0.0000000	0.00000%
6	0.9994206	0.9993654	0.9993642	-0.0000552	-0.0000565	0.0000012	0.00012%
7	0.9994206	0.9993778	0.9993776	-0.0000429	-0.0000430	0.0000001	0.00001%
8	0.9994206	0.9994071	0.9994069	-0.0000136	-0.0000138	0.0000002	0.00002%
9	0.9994206	0.9994087	0.9993872	-0.0000119	-0.0000334	0.0000214	0.00215%
10	0.9994206	0.9994041	0.9994037	-0.0000165	-0.0000169	0.0000004	0.00004%
11	0.9994206	0.9994123	0.9994123	-0.0000083	-0.0000083	0.0000000	0.00000%
12	0.9994206	0.9994178	0.9994178	-0.0000028	-0.0000028	0.0000000	0.00000%
13	0.9994206	0.9994249	0.9994249	0.0000043	0.0000043	0.0000000	0.00000%
14	0.9994206	0.9994190	0.9994190	-0.0000016	-0.0000016	0.0000000	0.00000%

(ii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9985563	0.9985340	0.9985337	-0.0000223	-0.0000226	0.0000003	0.00003%
2	0.9985563	0.9985094	0.9985082	-0.0000469	-0.0000481	0.0000012	0.00012%
3	0.9985563	0.9985548	0.9985547	-0.0000015	-0.0000016	0.0000001	0.00001%
4	0.9985563	0.9985154	0.9985154	-0.0000409	-0.0000409	0.0000000	0.00000%
5	0.9985563	0.9985749	0.9985749	0.0000187	0.0000186	0.0000000	0.00000%
6	0.9985563	0.9984948	0.9984944	-0.0000614	-0.0000619	0.0000005	0.00005%
7	0.9985563	0.9984991	0.9984988	-0.0000572	-0.0000574	0.0000002	0.00002%
8	0.9985563	0.9985689	0.9985684	0.0000127	0.0000121	0.0000006	0.00006%
9	0.9985563	0.9985260	0.9985131	-0.0000303	-0.0000431	0.0000128	0.00128%
10	0.9985563	0.9985486	0.9985483	-0.0000077	-0.0000080	0.0000004	0.00004%
11	0.9985563	0.9985482	0.9985482	-0.0000081	-0.0000081	0.0000000	0.00000%
12	0.9985563	0.9985510	0.9985510	-0.0000053	-0.0000053	0.0000000	0.00000%
13	0.9985563	0.9985610	0.9985610	0.0000047	0.0000047	0.0000000	0.00000%
14	0.9985563	0.9985571	0.9985571	0.0000009	0.0000009	0.0000000	0.00000%

(iii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9962834	0.9962303	0.9962298	-0.0000530	-0.0000535	0.0000005	0.00005%
2	0.9962834	0.9961500	0.9961471	-0.0001334	-0.0001363	0.0000029	0.00029%
3	0.9962834	0.9962115	0.9962105	-0.0000719	-0.0000729	0.0000010	0.00010%
4	0.9962834	0.9962185	0.9962183	-0.0000649	-0.0000651	0.0000002	0.00002%
5	0.9962834	0.9962773	0.9962771	-0.0000061	-0.0000063	0.0000002	0.00002%
6	0.9962834	0.9961764	0.9961764	-0.0001070	-0.0001070	0.0000000	0.00000%
7	0.9962834	0.9961883	0.9961877	-0.0000951	-0.0000957	0.0000005	0.00005%
8	0.9962834	0.9963521	0.9963508	0.0000687	0.0000674	0.0000013	0.00013%
9	0.9962834	0.9962487	0.9962425	-0.0000347	-0.0000409	0.0000062	0.00062%
10	0.9962834	0.9962886	0.9962883	0.0000052	0.0000049	0.0000003	0.00003%
11	0.9962834	0.9962750	0.9962750	-0.0000084	-0.0000084	0.0000000	0.00000%
12	0.9962834	0.9962715	0.9962715	-0.0000119	-0.0000119	0.0000000	0.00000%
13	0.9962834	0.9962867	0.9962867	0.0000033	0.0000033	0.0000000	0.00000%
14	0.9962834	0.9962895	0.9962895	0.0000062	0.0000062	0.0000000	0.00000%

Table 5.26 (continued) **–** Investigation of the accuracy of Approximation 1 for England and Wales female data – age in 2005 (iv) 65, (v) 75 – column (1) = 'deterministic' probability of surviving 2006, (2) = linear approximation to conditional probability of surviving 2006, (3) = actual conditional probability of surviving 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(iv)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9908891	0.9907113	0.9907095	-0.0001778	-0.0001797	0.0000019	0.00019%
2	0.9908891	0.9904959	0.9904869	-0.0003932	-0.0004022	0.0000090	0.00091%
3	0.9908891	0.9906073	0.9906026	-0.0002818	-0.0002865	0.0000047	0.00047%
4	0.9908891	0.9907928	0.9907925	-0.0000963	-0.0000966	0.0000003	0.00003%
5	0.9908891	0.9908061	0.9908055	-0.0000830	-0.0000837	0.0000007	0.00007%
6	0.9908891	0.9906266	0.9906253	-0.0002625	-0.0002639	0.0000014	0.00014%
7	0.9908891	0.9907205	0.9907194	-0.0001686	-0.0001697	0.0000011	0.00011%
8	0.9908891	0.9910552	0.9910531	0.0001660	0.0001640	0.0000020	0.00021%
9	0.9908891	0.9909036	0.9909022	0.0000145	0.0000131	0.0000014	0.00014%
10	0.9908891	0.9909085	0.9909083	0.0000194	0.0000192	0.0000002	0.00002%
11	0.9908891	0.9908813	0.9908813	-0.0000078	-0.0000078	0.0000000	0.00000%
12	0.9908891	0.9908620	0.9908619	-0.0000272	-0.0000272	0.0000000	0.00000%
13	0.9908891	0.9908800	0.9908800	-0.0000092	-0.0000092	0.0000000	0.00000%
14	0.9908891	0.9909027	0.9909027	0.0000136	0.0000136	0.0000000	0.00000%

(v)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.9730089	0.9721838	0.9721711	-0.0008252	-0.0008379	0.0000127	0.00131%
2	0.9730089	0.9715429	0.9715025	-0.0014660	-0.0015064	0.0000404	0.00416%
3	0.9730089	0.9719583	0.9719375	-0.0010506	-0.0010714	0.0000208	0.00214%
4	0.9730089	0.9729045	0.9729044	-0.0001044	-0.0001046	0.0000001	0.00001%
5	0.9730089	0.9726424	0.9726396	-0.0003665	-0.0003694	0.0000029	0.00029%
6	0.9730089	0.9721282	0.9721173	-0.0008807	-0.0008917	0.0000109	0.00112%
7	0.9730089	0.9726394	0.9726373	-0.0003696	-0.0003717	0.0000021	0.00022%
8	0.9730089	0.9733210	0.9733189	0.0003121	0.0003099	0.0000021	0.00022%
9	0.9730089	0.9730742	0.9730740	0.0000653	0.0000650	0.0000002	0.00003%
10	0.9730089	0.9730318	0.9730317	0.0000228	0.0000227	0.0000001	0.00001%
11	0.9730089	0.9730009	0.9730009	-0.0000081	-0.0000081	0.0000000	0.00000%
12	0.9730089	0.9729313	0.9729312	-0.0000776	-0.0000778	0.0000001	0.00001%
13	0.9730089	0.9729352	0.9729351	-0.0000738	-0.0000739	0.0000001	0.00001%
14	0.9730089	0.9730175	0.9730175	0.0000086	0.0000086	0.0000000	0.00000%

Table 5.27 – Investigation of the accuracy of Approximation 2 for CMI data – age in 2005 (i) 35, (ii) 45, (iii) 55 – column (1) = 'deterministic' mean annuity value at end of 2006, (2) = linear approximation to conditional mean annuity value at end of 2006, (3) = actual conditional mean annuity value at end of 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(i)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	4.38954	4.36616	4.36608	-0.02338	-0.02346	0.00008	0.0017%
2	4.38954	4.34827	4.34797	-0.04127	-0.04157	0.00029	0.0067%
3	4.38954	4.36363	4.36346	-0.02591	-0.02608	0.00017	0.0039%
4	4.38954	4.38721	4.38719	-0.00233	-0.00235	0.00002	0.0004%
5	4.38954	4.37909	4.37905	-0.01045	-0.01049	0.00004	0.0010%
6	4.38954	4.37794	4.37793	-0.01160	-0.01161	0.00001	0.0003%
7	4.38954	4.37931	4.37926	-0.01023	-0.01028	0.00005	0.0011%
8	4.38954	4.40163	4.40155	0.01209	0.01201	0.00008	0.0018%
9	4.38954	4.36003	4.35970	-0.02951	-0.02984	0.00033	0.0077%
10	4.38954	4.39538	4.39537	0.00584	0.00583	0.00002	0.0003%
11	4.38954	4.38945	4.38945	-0.00009	-0.00009	0.00000	0.0000%
12	4.38954	4.38723	4.38722	-0.00231	-0.00232	0.00000	0.0000%
13	4.38954	4.38885	4.38885	-0.00069	-0.00069	0.00000	0.0000%
14	4.38954	4.38964	4.38964	0.00010	0.00010	0.00000	0.0000%

(ii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	6.51325	6.47773	6.47768	-0.03552	-0.03556	0.00004	0.0007%
2	6.51325	6.44945	6.44918	-0.06379	-0.06406	0.00027	0.0042%
3	6.51325	6.46611	6.46579	-0.04714	-0.04745	0.00032	0.0049%
4	6.51325	6.51441	6.51439	0.00116	0.00114	0.00002	0.0003%
5	6.51325	6.49189	6.49180	-0.02135	-0.02144	0.00009	0.0014%
6	6.51325	6.49701	6.49701	-0.01623	-0.01623	0.00000	0.0000%
7	6.51325	6.50376	6.50374	-0.00948	-0.00951	0.00003	0.0004%
8	6.51325	6.53304	6.53286	0.01979	0.01961	0.00018	0.0027%
9	6.51325	6.48255	6.48243	-0.03070	-0.03082	0.00012	0.0019%
10	6.51325	6.52278	6.52276	0.00954	0.00951	0.00003	0.0004%
11	6.51325	6.51311	6.51311	-0.00014	-0.00014	0.00000	0.0000%
12	6.51325	6.50940	6.50940	-0.00385	-0.00385	0.00000	0.0000%
13	6.51325	6.51150	6.51150	-0.00175	-0.00175	0.00000	0.0000%
14	6.51325	6.51362	6.51362	0.00037	0.00037	0.00000	0.0000%

(iii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	9.63762	9.58712	9.58716	-0.05049	-0.05046	-0.00003	-0.0003%
2	9.63762	9.54618	9.54609	-0.09144	-0.09153	0.00009	0.0009%
3	9.63762	9.55252	9.55191	-0.08509	-0.08571	0.00062	0.0065%
4	9.63762	9.65246	9.65242	0.01484	0.01481	0.00003	0.0004%
5	9.63762	9.59406	9.59384	-0.04356	-0.04378	0.00022	0.0023%
6	9.63762	9.61768	9.61769	-0.01993	-0.01992	-0.00001	-0.0001%
7	9.63762	9.64107	9.64105	0.00346	0.00343	0.00002	0.0003%
8	9.63762	9.66625	9.66591	0.02863	0.02830	0.00034	0.0035%
9	9.63762	9.63313	9.63305	-0.00448	-0.00456	0.00008	0.0008%
10	9.63762	9.65163	9.65158	0.01402	0.01397	0.00005	0.0005%
11	9.63762	9.63741	9.63741	-0.00021	-0.00021	0.00000	0.0000%
12	9.63762	9.63145	9.63145	-0.00616	-0.00617	0.00000	0.0000%
13	9.63762	9.63318	9.63318	-0.00443	-0.00443	0.00000	0.0000%
14	9.63762	9.63846	9.63846	0.00085	0.00085	0.00000	0.0000%

Table 5.27 (continued) **–** Investigation of the accuracy of Approximation 2 for CMI data – age in 2005 (iv) 65, (v) 75 – column (1) = 'deterministic' mean annuity value at end of 2006, (2) = linear approximation to conditional mean annuity value at end of 2006, (3) = actual conditional mean annuity value at end of 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(iv)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	14.01838	13.94327	13.94344	-0.07510	-0.07493	-0.00017	-0.0012%
2	14.01838	13.89906	13.89931	-0.11931	-0.11906	-0.00025	-0.0018%
3	14.01838	13.88801	13.88709	-0.13037	-0.13128	0.00092	0.0066%
4	14.01838	14.06519	14.06506	0.04681	0.04668	0.00013	0.0009%
5	14.01838	13.94582	13.94541	-0.07255	-0.07297	0.00042	0.0030%
6	14.01838	13.99219	13.99221	-0.02618	-0.02616	-0.00002	-0.0001%
7	14.01838	14.05461	14.05440	0.03623	0.03603	0.00021	0.0015%
8	14.01838	14.03529	14.03498	0.01691	0.01661	0.00030	0.0022%
9	14.01838	14.07293	14.07174	0.05455	0.05337	0.00118	0.0084%
10	14.01838	14.03039	14.03035	0.01202	0.01197	0.00005	0.0003%
11	14.01838	14.01815	14.01815	-0.00023	-0.00023	0.00000	0.0000%
12	14.01838	14.00985	14.00985	-0.00852	-0.00853	0.00000	0.0000%
13	14.01838	14.00821	14.00821	-0.01017	-0.01017	0.00000	0.0000%
14	14.01838	14.01779	14.01779	-0.00058	-0.00059	0.00000	0.0000%

(v)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	10.17410	10.04807	10.04868	-0.12603	-0.12542	-0.00061	-0.0061%
2	10.17410	10.02132	10.02215	-0.15277	-0.15195	-0.00083	-0.0082%
3	10.17410	10.02491	10.02438	-0.14919	-0.14972	0.00053	0.0053%
4	10.17410	10.25984	10.25963	0.08574	0.08554	0.00020	0.0020%
5	10.17410	10.09007	10.08975	-0.08402	-0.08435	0.00032	0.0032%
6	10.17410	10.12492	10.12505	-0.04918	-0.04904	-0.00014	-0.0014%
7	10.17410	10.24239	10.24200	0.06829	0.06791	0.00038	0.0037%
8	10.17410	10.13223	10.13226	-0.04187	-0.04184	-0.00003	-0.0003%
9	10.17410	10.25606	10.25430	0.08196	0.08020	0.00176	0.0171%
10	10.17410	10.16868	10.16867	-0.00542	-0.00543	0.00001	0.0001%
11	10.17410	10.17395	10.17395	-0.00015	-0.00015	0.00000	0.0000%
12	10.17410	10.16454	10.16454	-0.00956	-0.00956	0.00000	0.0000%
13	10.17410	10.15617	10.15617	-0.01793	-0.01793	0.00000	0.0000%
14	10.17410	10.16685	10.16685	-0.00725	-0.00725	0.00000	0.0000%

Table 5.28 – Investigation of the accuracy of Approximation 2 for England and Wales male data – age in 2005 (i) 35, (ii) 45, (iii) 55 – column (1) = 'deterministic' mean annuity value at end of 2006, (2) = linear approximation to conditional mean annuity value at end of 2006, (3) = actual conditional mean annuity value at end of 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(i)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	3.52276	3.49756	3.49751	-0.02521	-0.02525	0.00005	0.0013%
2	3.52276	3.47717	3.47697	-0.04559	-0.04580	0.00021	0.0060%
3	3.52276	3.49719	3.49710	-0.02558	-0.02567	0.00009	0.0025%
4	3.52276	3.51767	3.51766	-0.00509	-0.00510	0.00001	0.0003%
5	3.52276	3.51297	3.51296	-0.00979	-0.00981	0.00002	0.0005%
6	3.52276	3.51260	3.51260	-0.01016	-0.01016	0.00000	0.0001%
7	3.52276	3.50835	3.50832	-0.01441	-0.01445	0.00003	0.0009%
8	3.52276	3.53829	3.53823	0.01552	0.01547	0.00006	0.0016%
9	3.52276	3.47718	3.47681	-0.04558	-0.04596	0.00037	0.0107%
10	3.52276	3.53076	3.53074	0.00799	0.00798	0.00001	0.0004%
11	3.52276	3.52250	3.52250	-0.00026	-0.00026	0.00000	0.0000%
12	3.52276	3.52016	3.52016	-0.00260	-0.00260	0.00000	0.0000%
13	3.52276	3.52247	3.52247	-0.00029	-0.00029	0.00000	0.0000%
14	3.52276	3.52298	3.52298	0.00022	0.00022	0.00000	0.0000%

(ii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	5.27308	5.23410	5.23407	-0.03897	-0.03901	0.00003	0.0006%
2	5.27308	5.20116	5.20091	-0.07191	-0.07216	0.00025	0.0048%
3	5.27308	5.22883	5.22868	-0.04425	-0.04439	0.00014	0.0027%
4	5.27308	5.26651	5.26649	-0.00657	-0.00658	0.00002	0.0003%
5	5.27308	5.25532	5.25529	-0.01775	-0.01778	0.00003	0.0005%
6	5.27308	5.25576	5.25575	-0.01731	-0.01732	0.00001	0.0002%
7	5.27308	5.25260	5.25256	-0.02047	-0.02052	0.00004	0.0008%
8	5.27308	5.29854	5.29843	0.02546	0.02535	0.00011	0.0020%
9	5.27308	5.21260	5.21242	-0.06048	-0.06065	0.00018	0.0034%
10	5.27308	5.28536	5.28534	0.01229	0.01227	0.00002	0.0003%
11	5.27308	5.27272	5.27272	-0.00035	-0.00035	0.00000	0.0000%
12	5.27308	5.26900	5.26899	-0.00408	-0.00408	0.00000	0.0000%
13	5.27308	5.27229	5.27229	-0.00079	-0.00079	0.00000	0.0000%
14	5.27308	5.27366	5.27366	0.00059	0.00058	0.00000	0.0000%

(iii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	8.02286	7.96338	7.96339	-0.05949	-0.05948	-0.00001	-0.0001%
2	8.02286	7.91255	7.91229	-0.11032	-0.11057	0.00026	0.0032%
3	8.02286	7.94724	7.94699	-0.07563	-0.07588	0.00025	0.0032%
4	8.02286	8.01762	8.01760	-0.00524	-0.00526	0.00002	0.0002%
5	8.02286	7.99065	7.99060	-0.03222	-0.03226	0.00005	0.0006%
6	8.02286	7.99425	7.99422	-0.02862	-0.02865	0.00003	0.0004%
7	8.02286	7.99790	7.99786	-0.02496	-0.02500	0.00004	0.0005%
8	8.02286	8.06138	8.06121	0.03852	0.03834	0.00018	0.0022%
9	8.02286	7.95332	7.95346	-0.06954	-0.06940	-0.00014	-0.0017%
10	8.02286	8.04057	8.04056	0.01771	0.01769	0.00002	0.0002%
11	8.02286	8.02243	8.02243	-0.00044	-0.00044	0.00000	0.0000%
12	8.02286	8.01674	8.01674	-0.00613	-0.00613	0.00000	0.0000%
13	8.02286	8.02078	8.02078	-0.00208	-0.00208	0.00000	0.0000%
14	8.02286	8.02395	8.02395	0.00108	0.00108	0.00000	0.0000%

Table 5.28 (continued) **–** Investigation of the accuracy of Approximation 2 for England and Wales male data – age in 2005 (iv) 65, (v) 75 – column (1) = 'deterministic' mean annuity value at end of 2006, (2) = linear approximation to conditional mean annuity value at end of 2006, (3) = actual conditional mean annuity value at end of 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(iv)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	12.24807	12.16089	12.16101	-0.08718	-0.08706	-0.00012	-0.0010%
2	12.24807	12.09734	12.09727	-0.15073	-0.15080	0.00007	0.0006%
3	12.24807	12.13780	12.13748	-0.11027	-0.11059	0.00032	0.0026%
4	12.24807	12.25522	12.25522	0.00715	0.00715	0.00000	0.0000%
5	12.24807	12.19716	12.19710	-0.05091	-0.05097	0.00006	0.0005%
6	12.24807	12.21214	12.21209	-0.03593	-0.03598	0.00005	0.0004%
7	12.24807	12.22993	12.22992	-0.01814	-0.01816	0.00002	0.0001%
8	12.24807	12.28888	12.28871	0.04081	0.04063	0.00017	0.0014%
9	12.24807	12.17213	12.17255	-0.07594	-0.07552	-0.00042	-0.0035%
10	12.24807	12.26986	12.26986	0.02179	0.02178	0.00001	0.0001%
11	12.24807	12.24771	12.24771	-0.00036	-0.00036	0.00000	0.0000%
12	12.24807	12.24028	12.24028	-0.00779	-0.00779	0.00000	0.0000%
13	12.24807	12.24339	12.24339	-0.00468	-0.00468	0.00000	0.0000%
14	12.24807	12.24827	12.24827	0.00020	0.00020	0.00000	0.0000%

(v)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	8.49986	8.38437	8.38479	-0.11549	-0.11507	-0.00042	-0.0050%
2	8.49986	8.32729	8.32791	-0.17258	-0.17196	-0.00062	-0.0075%
3	8.49986	8.37758	8.37758	-0.12228	-0.12229	0.00000	0.0000%
4	8.49986	8.53047	8.53050	0.03061	0.03063	-0.00003	-0.0003%
5	8.49986	8.43822	8.43823	-0.06164	-0.06164	0.00000	0.0000%
6	8.49986	8.47137	8.47135	-0.02849	-0.02852	0.00002	0.0003%
7	8.49986	8.50118	8.50118	0.00132	0.00131	0.00000	0.0000%
8	8.49986	8.51804	8.51798	0.01817	0.01812	0.00005	0.0006%
9	8.49986	8.39840	8.39925	-0.10147	-0.10062	-0.00085	-0.0101%
10	8.49986	8.52151	8.52151	0.02164	0.02165	-0.00001	-0.0001%
11	8.49986	8.49967	8.49967	-0.00020	-0.00020	0.00000	0.0000%
12	8.49986	8.49204	8.49204	-0.00783	-0.00782	0.00000	0.0000%
13	8.49986	8.49230	8.49230	-0.00756	-0.00756	0.00000	0.0000%
14	8.49986	8.49635	8.49635	-0.00351	-0.00351	0.00000	0.0000%

Table 5.29 – Investigation of the accuracy of Approximation 2 for England and Wales female data – age in 2005 (i) 35, (ii) 45, (iii) 55 – column (1) = 'deterministic' mean annuity value at end of 2006, (2) = linear approximation to conditional mean annuity value at end of 2006, (3) = actual conditional mean annuity value at end of 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(i)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	3.71396	3.68773	3.68766	-0.02623	-0.02630	0.00006	0.0017%
2	3.71396	3.66715	3.66689	-0.04681	-0.04707	0.00026	0.0070%
3	3.71396	3.68868	3.68859	-0.02528	-0.02537	0.00009	0.0025%
4	3.71396	3.70832	3.70832	-0.00563	-0.00564	0.00000	0.0001%
5	3.71396	3.70467	3.70466	-0.00929	-0.00930	0.00002	0.0005%
6	3.71396	3.70277	3.70275	-0.01119	-0.01121	0.00002	0.0007%
7	3.71396	3.69852	3.69850	-0.01544	-0.01546	0.00003	0.0007%
8	3.71396	3.72908	3.72901	0.01512	0.01505	0.00007	0.0019%
9	3.71396	3.66623	3.66538	-0.04773	-0.04858	0.00085	0.0233%
10	3.71396	3.72176	3.72173	0.00780	0.00777	0.00003	0.0008%
11	3.71396	3.71350	3.71350	-0.00046	-0.00046	0.00000	0.0000%
12	3.71396	3.71108	3.71108	-0.00287	-0.00288	0.00000	0.0000%
13	3.71396	3.71352	3.71352	-0.00044	-0.00044	0.00000	0.0000%
14	3.71396	3.71410	3.71410	0.00014	0.00014	0.00000	0.0000%

(ii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	5.70037	5.66011	5.66004	-0.04027	-0.04034	0.00007	0.0012%
2	5.70037	5.62787	5.62752	-0.07251	-0.07285	0.00034	0.0061%
3	5.70037	5.65840	5.65825	-0.04197	-0.04212	0.00015	0.0027%
4	5.70037	5.69207	5.69206	-0.00831	-0.00832	0.00001	0.0002%
5	5.70037	5.68508	5.68505	-0.01529	-0.01532	0.00003	0.0005%
6	5.70037	5.67817	5.67816	-0.02221	-0.02222	0.00001	0.0002%
7	5.70037	5.67736	5.67732	-0.02302	-0.02305	0.00004	0.0007%
8	5.70037	5.72308	5.72297	0.02271	0.02259	0.00011	0.0020%
9	5.70037	5.64150	5.64091	-0.05888	-0.05947	0.00059	0.0105%
10	5.70037	5.71060	5.71057	0.01023	0.01020	0.00003	0.0005%
11	5.70037	5.69976	5.69976	-0.00062	-0.00062	0.00000	0.0000%
12	5.70037	5.69601	5.69601	-0.00437	-0.00437	0.00000	0.0000%
13	5.70037	5.69915	5.69915	-0.00122	-0.00122	0.00000	0.0000%
14	5.70037	5.70074	5.70074	0.00037	0.00037	0.00000	0.0000%

(iii)

Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	8.75036	8.68769	8.68763	-0.06268	-0.06273	0.00006	0.0007%
2	8.75036	8.63810	8.63768	-0.11226	-0.11268	0.00042	0.0049%
3	8.75036	8.68048	8.68023	-0.06988	-0.07013	0.00025	0.0029%
4	8.75036	8.73936	8.73934	-0.01101	-0.01102	0.00002	0.0002%
5	8.75036	8.72508	8.72504	-0.02528	-0.02532	0.00004	0.0005%
6	8.75036	8.70649	8.70647	-0.04388	-0.04389	0.00002	0.0002%
7	8.75036	8.71739	8.71734	-0.03297	-0.03302	0.00005	0.0006%
8	8.75036	8.78207	8.78192	0.03171	0.03155	0.00016	0.0018%
9	8.75036	8.68809	8.68791	-0.06227	-0.06245	0.00018	0.0021%
10	8.75036	8.76208	8.76205	0.01171	0.01169	0.00002	0.0003%
11	8.75036	8.74958	8.74958	-0.00078	-0.00078	0.00000	0.0000%
12	8.75036	8.74388	8.74387	-0.00649	-0.00649	0.00000	0.0000%
13	8.75036	8.74728	8.74728	-0.00309	-0.00309	0.00000	0.0000%
14	8.75036	8.75094	8.75094	0.00058	0.00058	0.00000	0.0000%

Table 5.29 (continued) **–** Investigation of the accuracy of Approximation 2 for England and Wales female data – age in 2005 (iv) 65, (v) 75 – column (1) = 'deterministic' mean annuity value at end of 2006, (2) = linear approximation to conditional mean annuity value at end of 2006, (3) = actual conditional mean annuity value at end of 2006, (4) = (2) – (1), (5) = (3) – (1), (6) = absolute error in approximation of (3) by (2), (7) = percentage error in approximation of (3) by (2)

(iv)

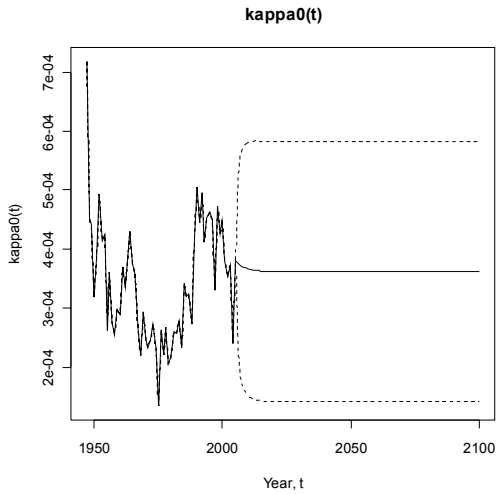
Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	13.30001	13.20689	13.20689	-0.09312	-0.09312	0.00000	0.0000%
2	13.30001	13.14210	13.14176	-0.15790	-0.15825	0.00035	0.0026%
3	13.30001	13.19748	13.19717	-0.10252	-0.10284	0.00032	0.0024%
4	13.30001	13.29238	13.29236	-0.00763	-0.00764	0.00001	0.0001%
5	13.30001	13.26237	13.26232	-0.03764	-0.03769	0.00005	0.0004%
6	13.30001	13.22828	13.22818	-0.07173	-0.07183	0.00010	0.0008%
7	13.30001	13.26219	13.26214	-0.03782	-0.03786	0.00005	0.0003%
8	13.30001	13.33162	13.33149	0.03161	0.03148	0.00013	0.0010%
9	13.30001	13.24541	13.24558	-0.05459	-0.05442	-0.00017	-0.0013%
10	13.30001	13.30972	13.30972	0.00972	0.00971	0.00001	0.0001%
11	13.30001	13.29936	13.29936	-0.00065	-0.00065	0.00000	0.0000%
12	13.30001	13.29176	13.29176	-0.00824	-0.00824	0.00000	0.0000%
13	13.30001	13.29343	13.29343	-0.00657	-0.00657	0.00000	0.0000%
14	13.30001	13.29962	13.29962	-0.00038	-0.00038	0.00000	0.0000%

(v)

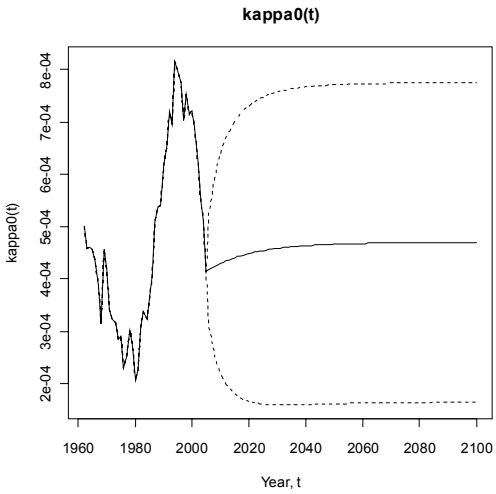
Test	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	9.40787	9.28182	9.28209	-0.12606	-0.12579	-0.00027	-0.0029%
2	9.40787	9.21288	9.21324	-0.19499	-0.19463	-0.00036	-0.0039%
3	9.40787	9.28254	9.28248	-0.12533	-0.12540	0.00006	0.0007%
4	9.40787	9.41208	9.41208	0.00421	0.00421	0.00000	0.0000%
5	9.40787	9.36082	9.36081	-0.04706	-0.04706	0.00001	0.0001%
6	9.40787	9.31408	9.31402	-0.09379	-0.09385	0.00006	0.0007%
7	9.40787	9.37452	9.37452	-0.03335	-0.03336	0.00000	0.0000%
8	9.40787	9.42309	9.42305	0.01522	0.01518	0.00004	0.0004%
9	9.40787	9.35774	9.35802	-0.05013	-0.04985	-0.00028	-0.0030%
10	9.40787	9.41204	9.41204	0.00417	0.00417	0.00000	0.0000%
11	9.40787	9.40751	9.40751	-0.00037	-0.00037	0.00000	0.0000%
12	9.40787	9.39914	9.39914	-0.00873	-0.00873	0.00000	0.0000%
13	9.40787	9.39715	9.39715	-0.01072	-0.01072	0.00000	0.0000%
14	9.40787	9.40452	9.40452	-0.00335	-0.00335	0.00000	0.0000%

Figure 5.1 – Projection of the future $\kappa^{(0)}$ parameters **–** solid curve = median, dashed curves = 95% prediction limits (2.5^{th} and 97.5^{th} percentiles) – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

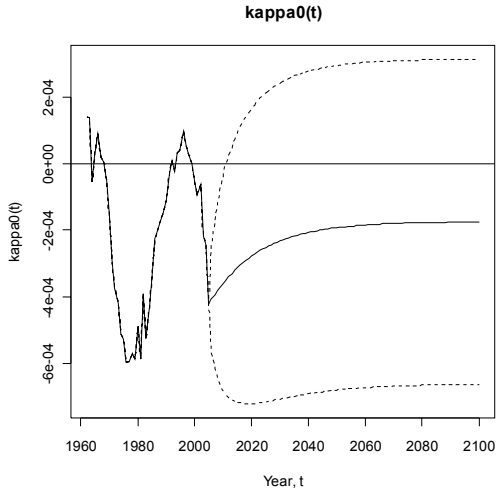
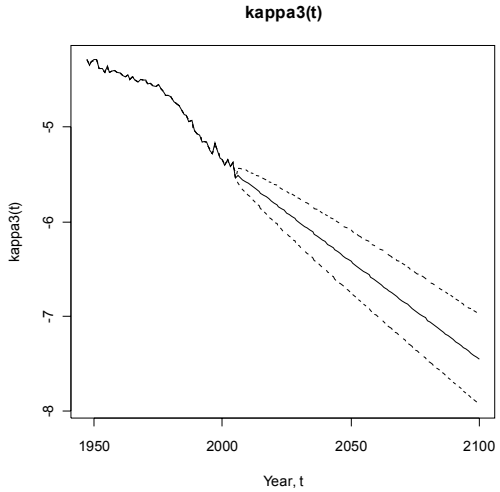
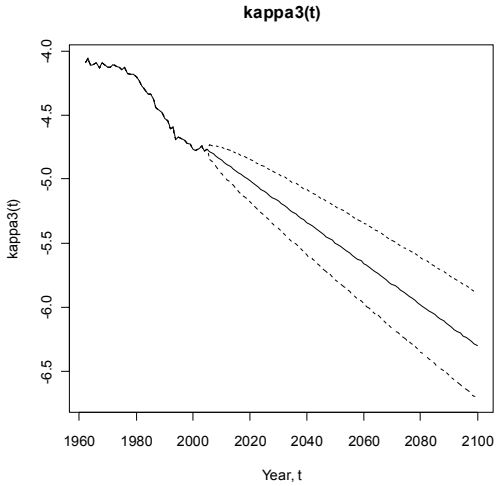


Figure 5.2 – Projection of the future $\kappa^{(3)}$ parameters **–** solid curve = median, dashed curves = 95% prediction limits (2.5^{th} and 97.5^{th} percentiles) – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

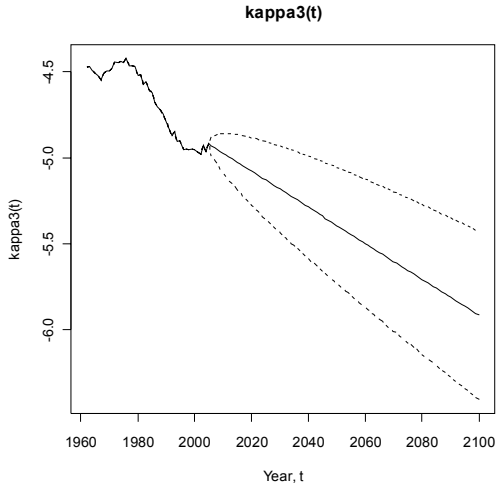
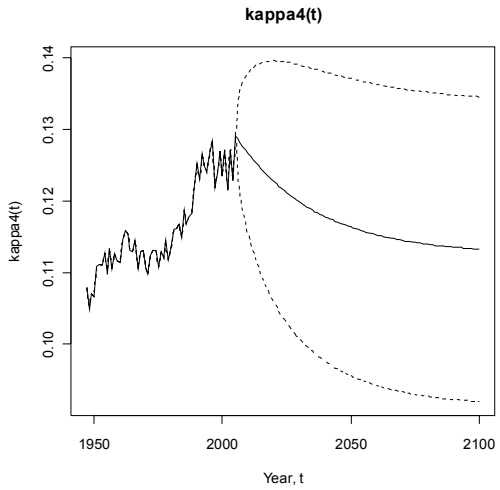
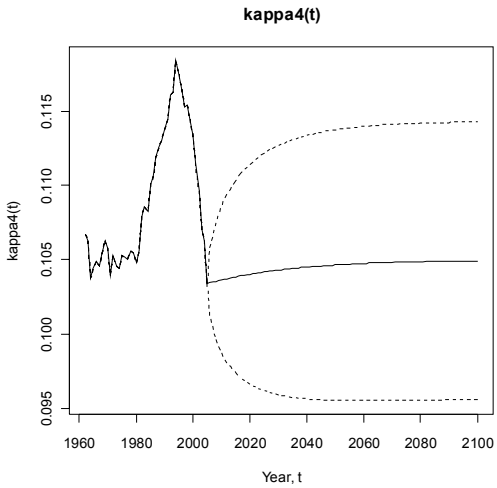


Figure 5.3 – Projection of the future $\kappa^{(4)}$ parameters **–** solid curve = median, dashed curves = 95% prediction limits (2.5^{th} and 97.5^{th} percentiles) – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

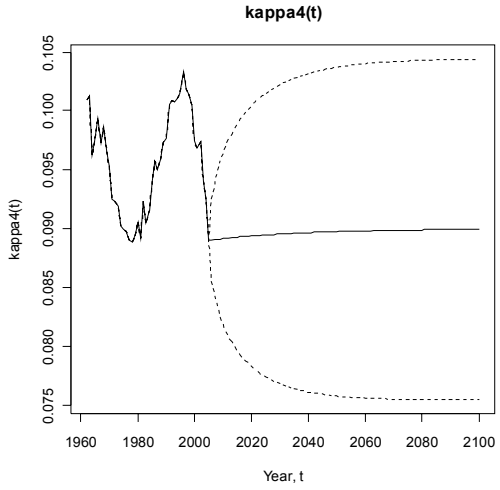
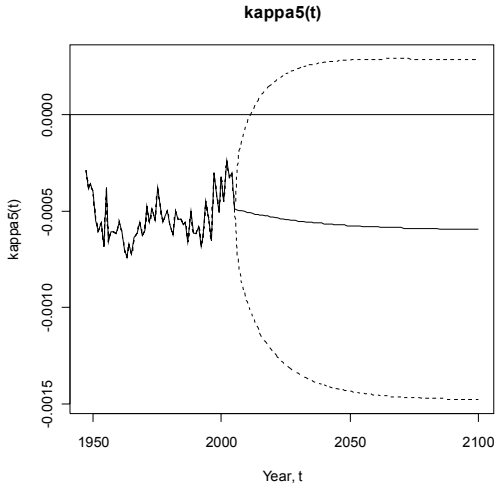
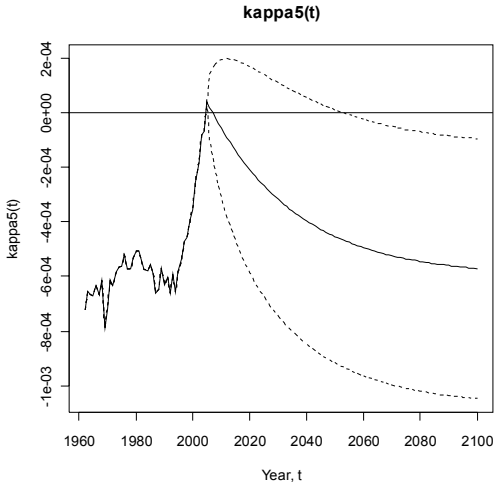


Figure 5.4 – Projection of the future $\kappa^{(5)}$ parameters **–** solid curve = median, dashed curves = 95% prediction limits (2.5^{th} and 97.5^{th} percentiles) – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

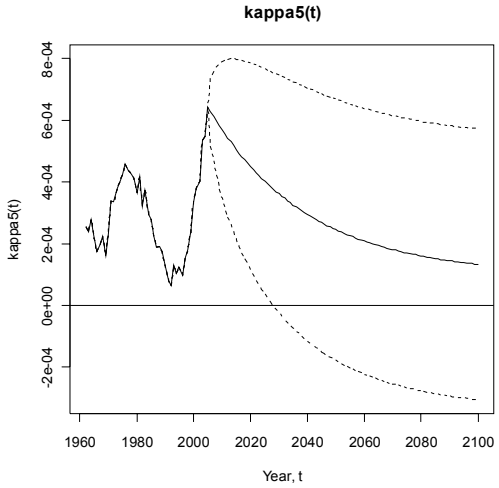
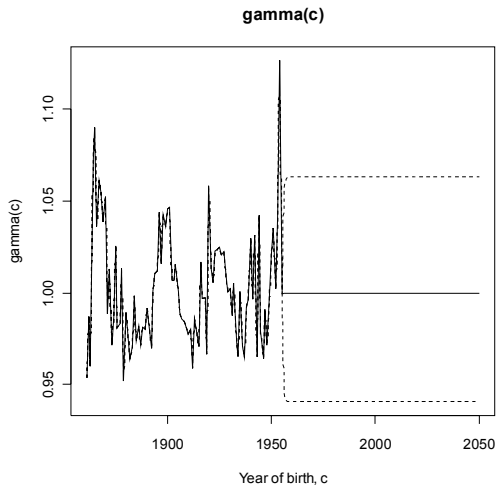
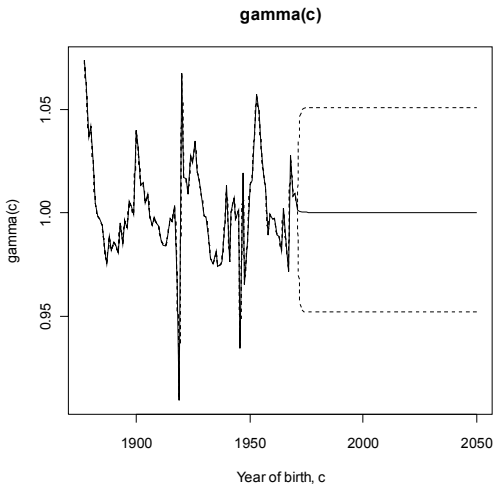


Figure 5.5 – Projection of the future γ parameters **–** solid curve = median, dashed curves = 95% prediction limits (2.5^{th} and 97.5^{th} percentiles) – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

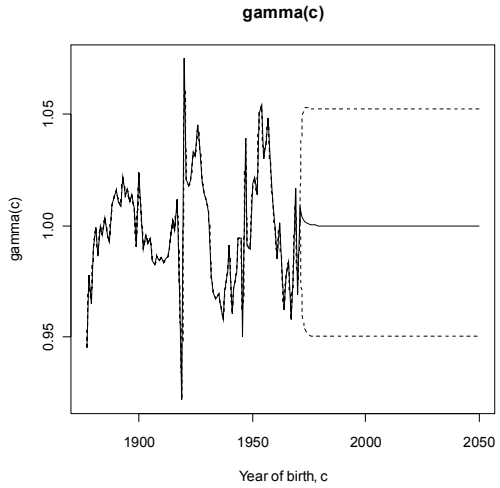
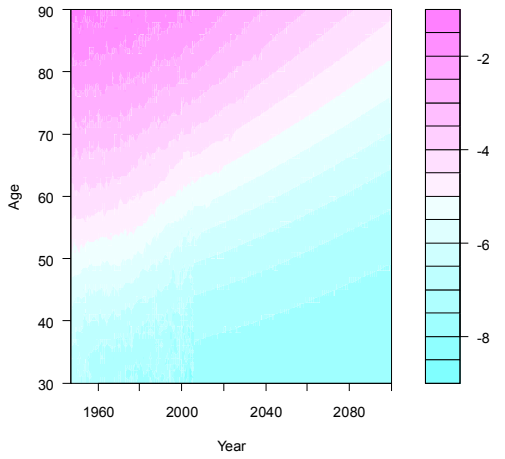
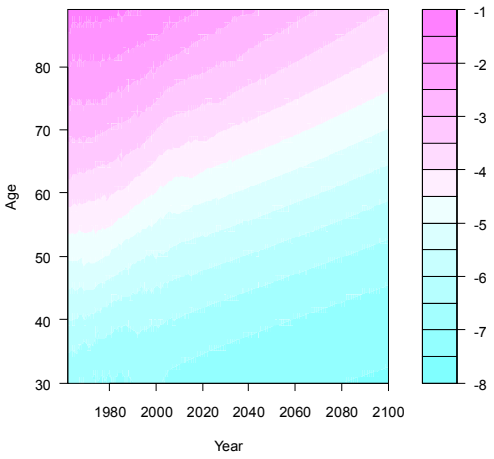


Figure 5.6 – Extensions of the graphs in Figure 1.2 to future years, using a deterministic projection with all the future innovation terms set to zero **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

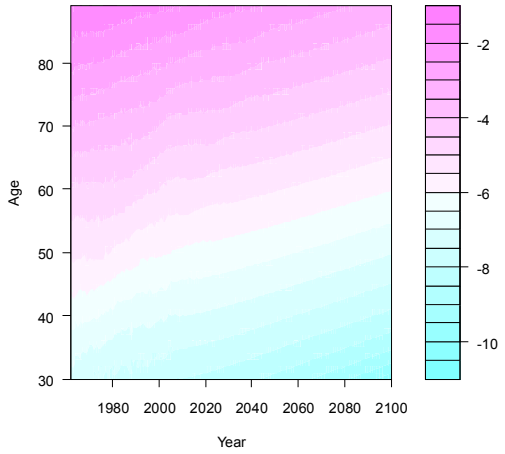
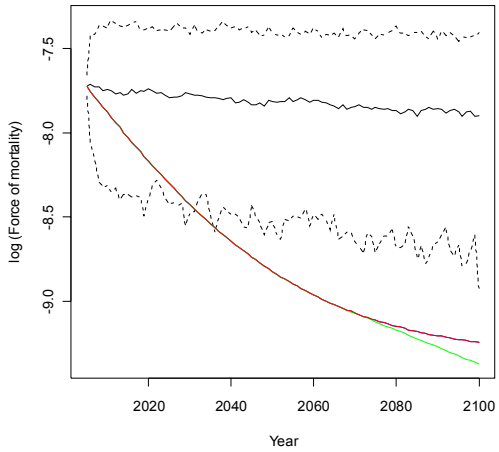
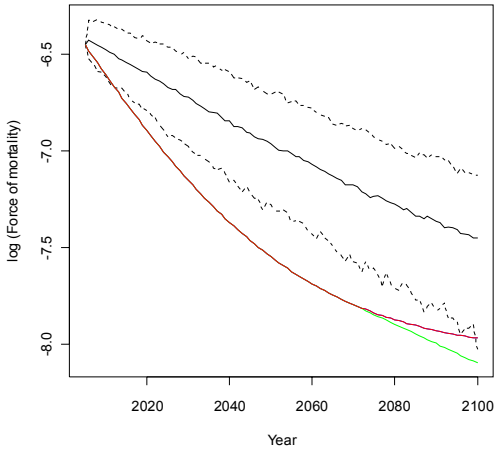


Figure 5.7 – Projected forces of mortality for CMI data compared with previously published projection bases – age (i) 30, (ii) 50, (iii) 70, (iv) 90. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = Medium Cohort, green solid curve = Medium Cohort subject to a 1% underpin, red solid curve = Long Cohort.

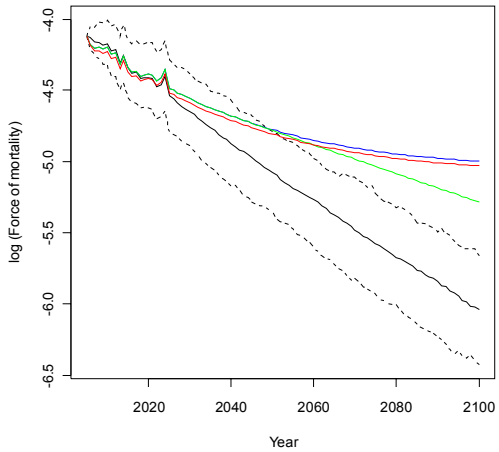
(i)



(ii)



(iii)



(iv)

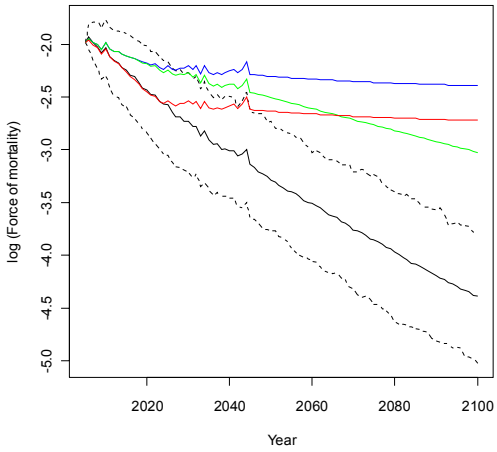
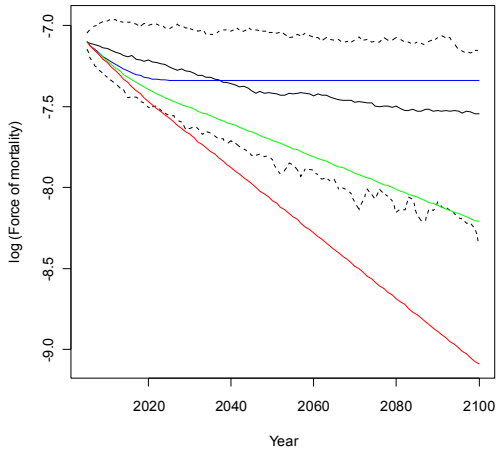
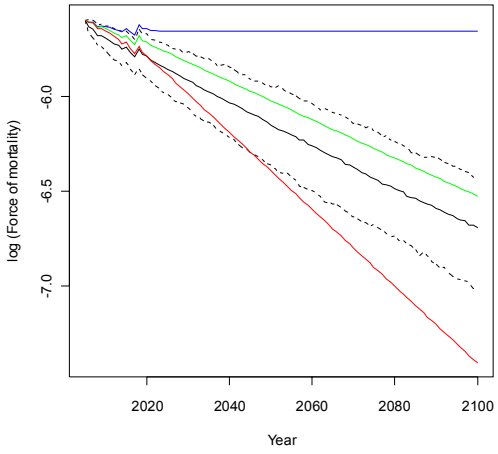


Figure 5.8 – Projected forces of mortality for England and Wales male data compared with GAD projection bases – age (i) 30, (ii) 50, (iii) 70, (iv) 89. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

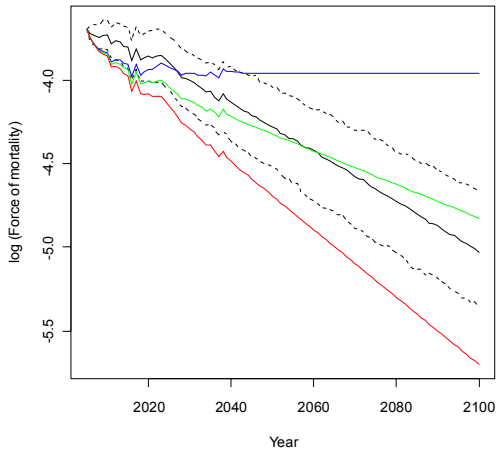
(i)



(ii)



(iii)



(iv)

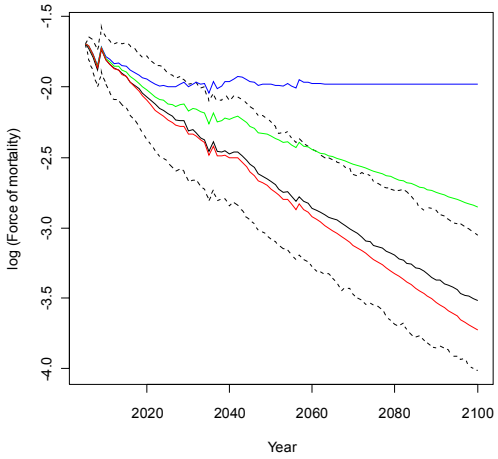
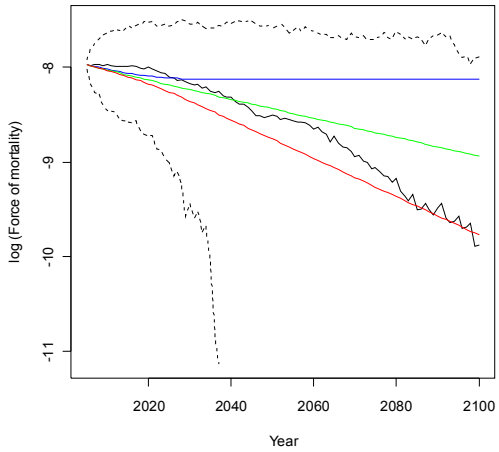
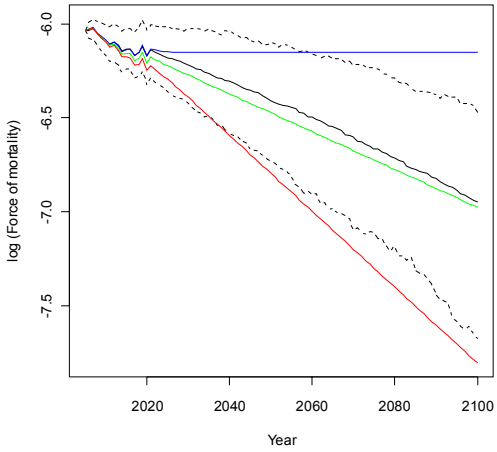


Figure 5.9 – Projected forces of mortality for England and Wales female data compared with GAD projection bases – age (i) 30, (ii) 50, (iii) 70, (iv) 89. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

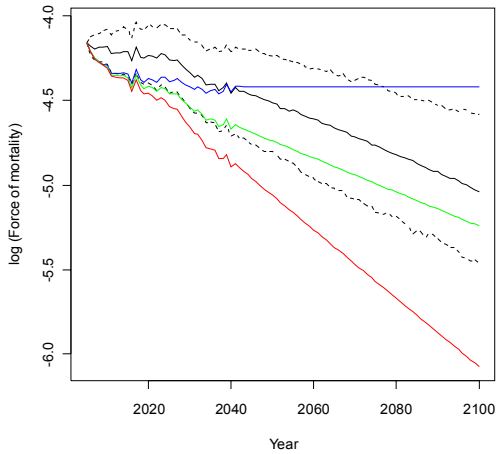
(i)



(ii)



(iii)



(iv)

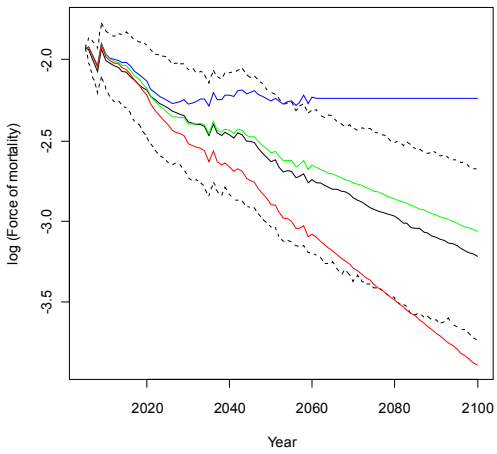
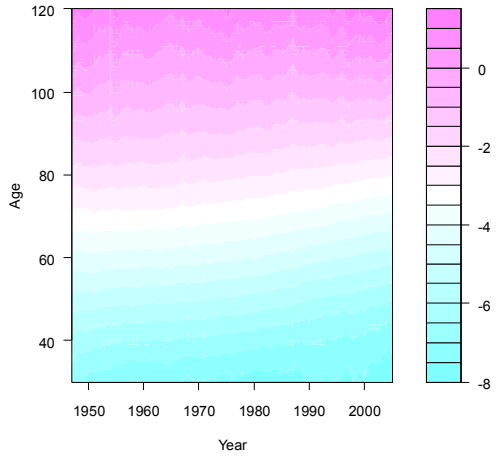
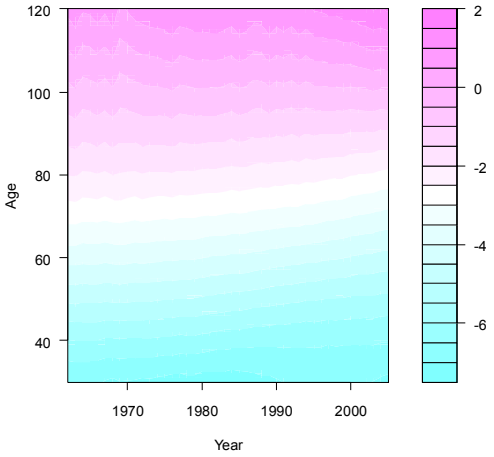


Figure 5.10 – Extrapolation of historical mortality to ages above 90 (89 for England and Wales data) using the log-linear extrapolation method, before the introduction of gamma parameters **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

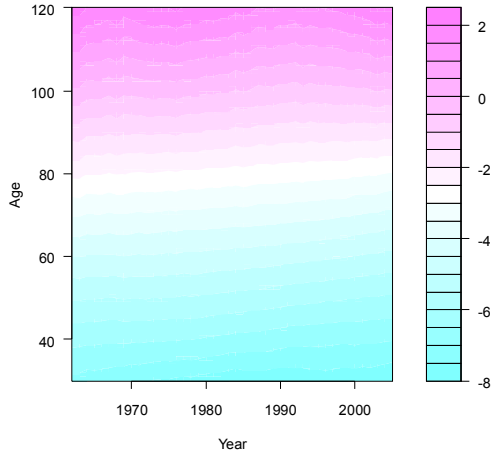
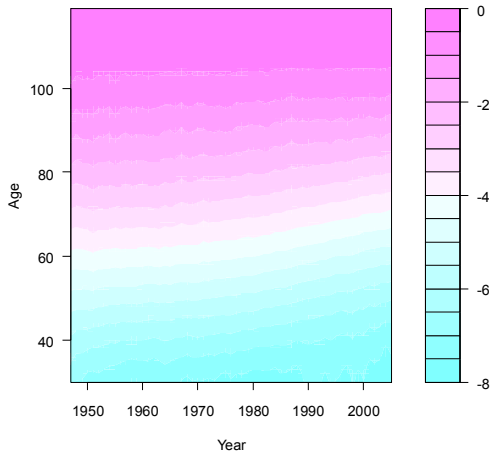
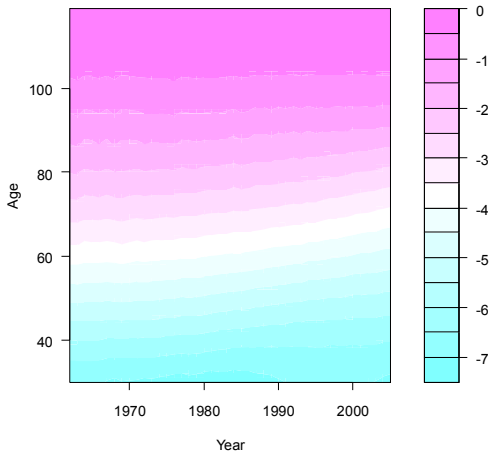


Figure 5.11 – Extrapolation of historical mortality to ages above 90 (89 for England and Wales data) using the LifeMetrics extrapolation method, before the introduction of gamma parameters **–** (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

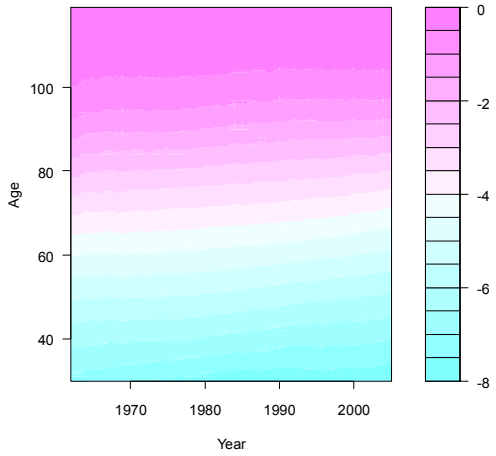
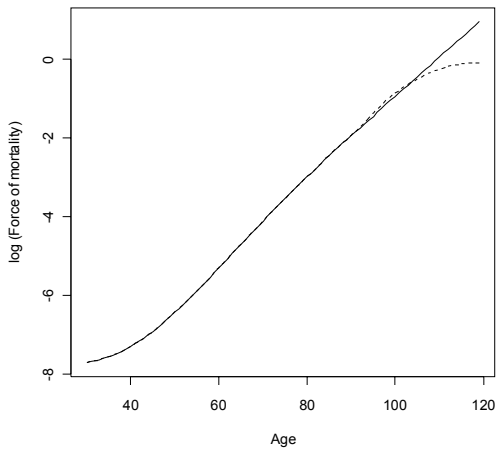
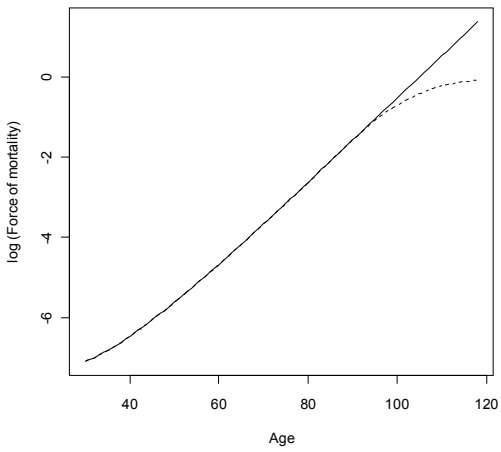


Figure 5.12 – Comparison of methods of extrapolating the 2005 mortality curve to ages above 90 (89 for England and Wales data), before the introduction of gamma parameters **–** solid curve = log-linear extrapolation method, dashed curve = LifeMetrics extrapolation method – (i) CMI data, (ii) England and Wales male data, (iii) England and Wales female data

(i)



(ii)



(iii)

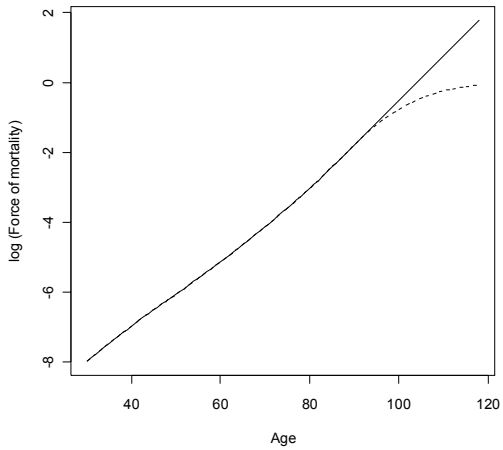
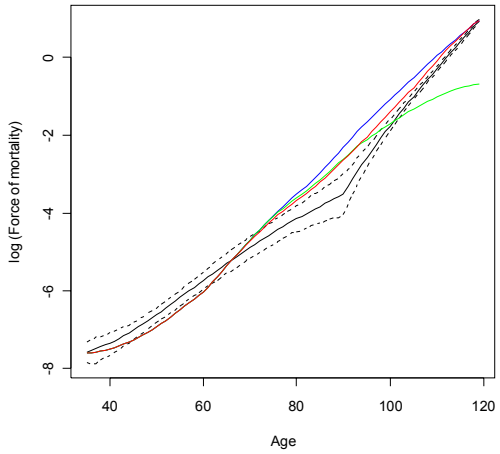
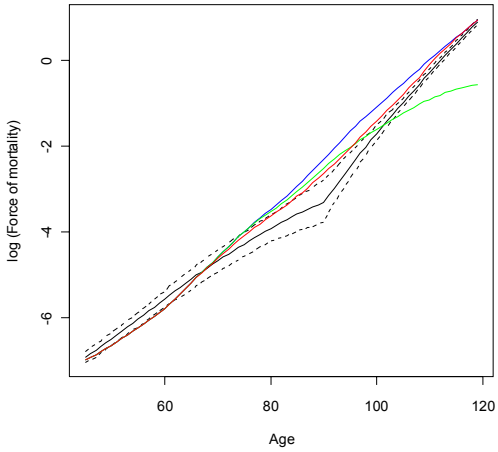


Figure 5.13 – Projected forces of mortality for CMI data under the low improvement assumption compared with previously published projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = Medium Cohort, green solid curve = Medium Cohort subject to a 1% underpin, red solid curve = Long Cohort.

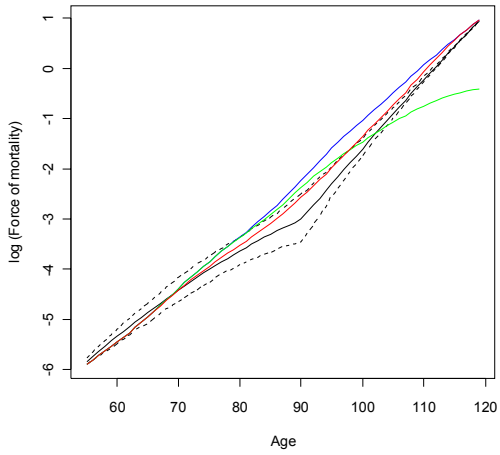
(i)



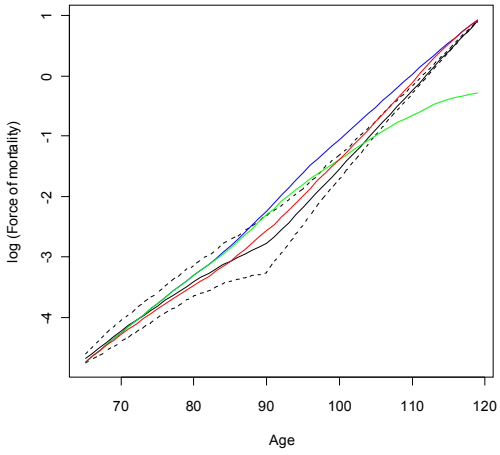
(ii)



(iii)



(iv)



(v)

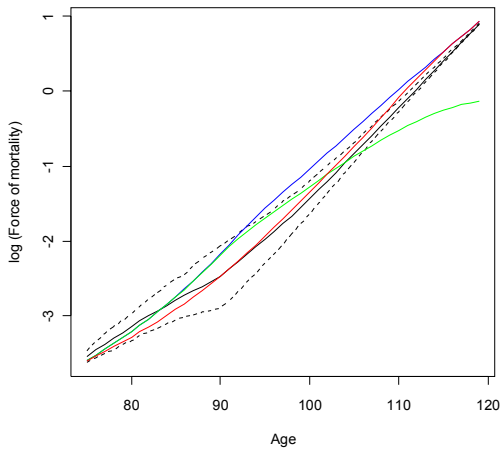
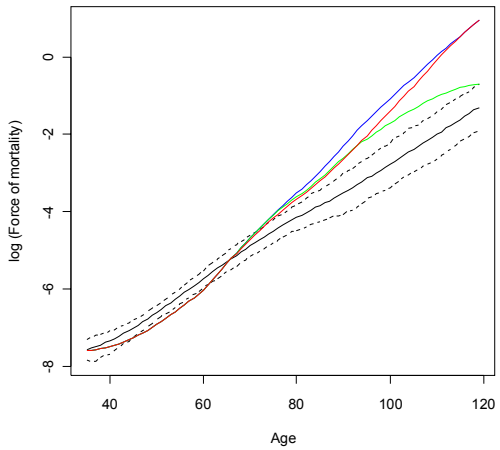
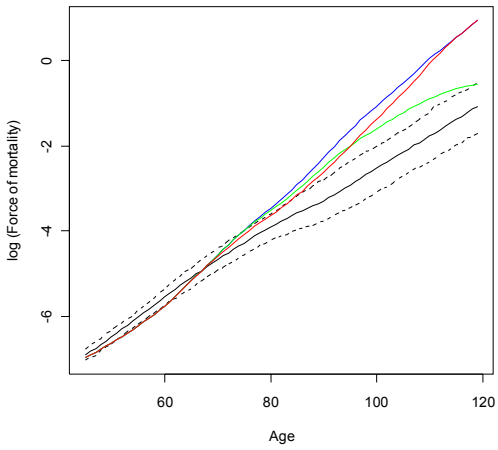


Figure 5.14 – Projected forces of mortality for CMI data under the high improvement assumption compared with previously published projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = Medium Cohort, green solid curve = Medium Cohort subject to a 1% underpin, red solid curve = Long Cohort.

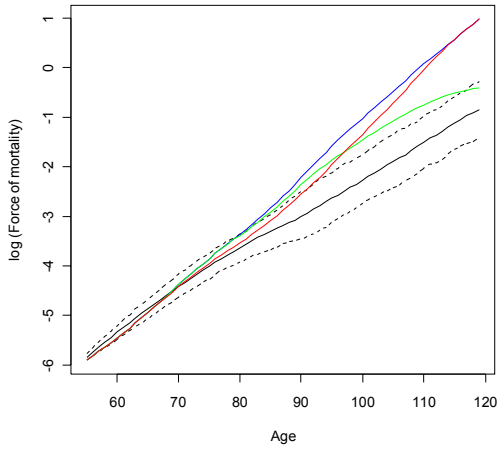
(i)



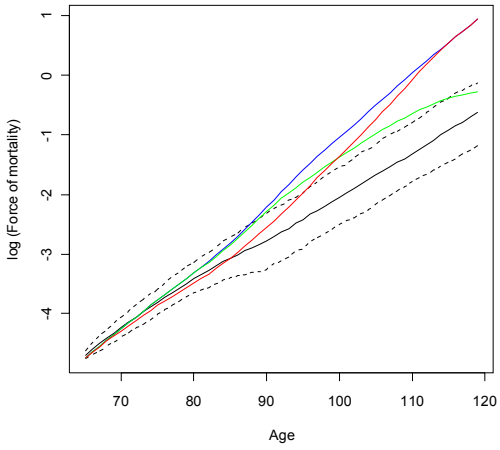
(ii)



(iii)



(iv)



(v)

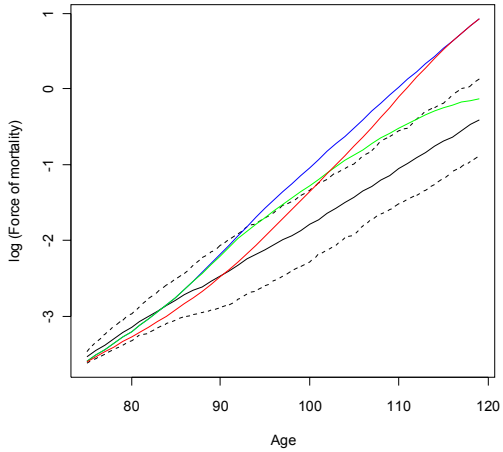
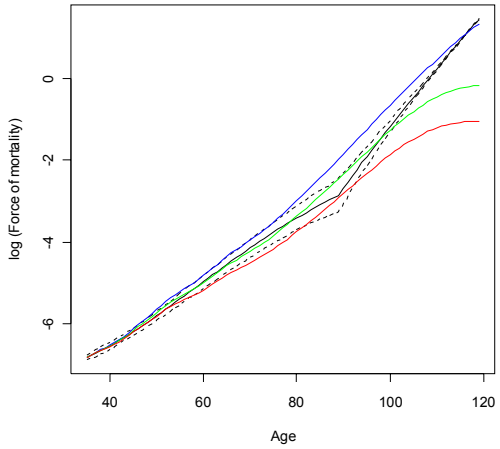
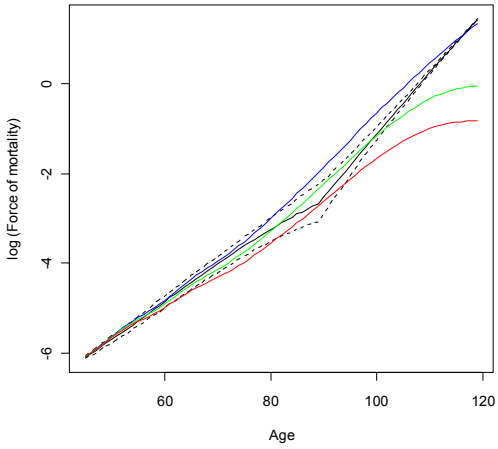


Figure 5.15 – Projected forces of mortality for England and Wales male data under the low improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

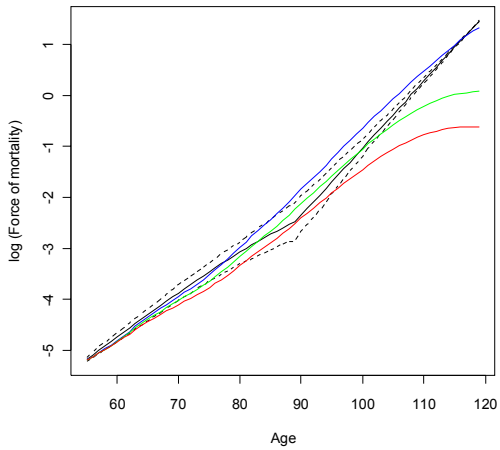
(i)



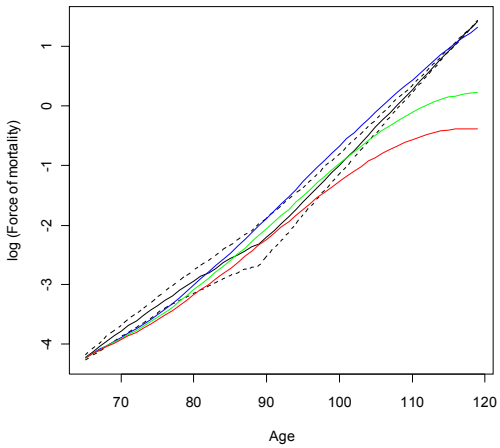
(ii)



(iii)



(iv)



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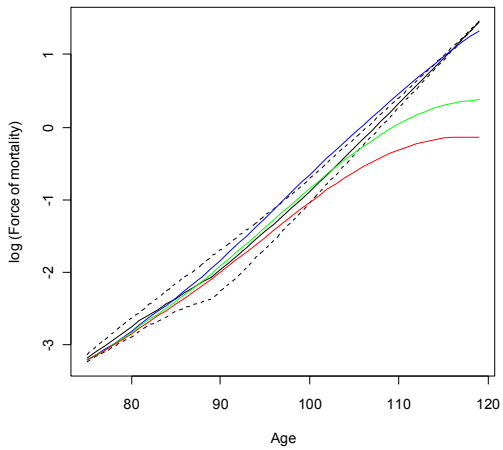
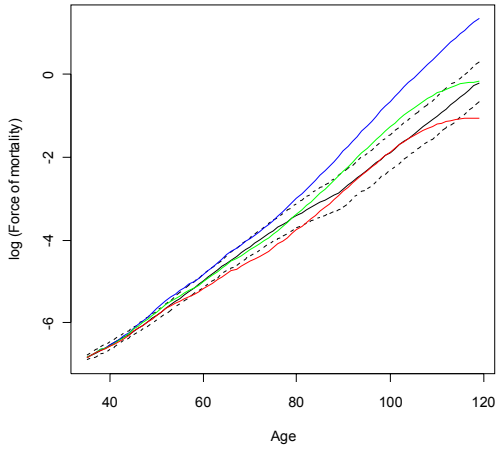
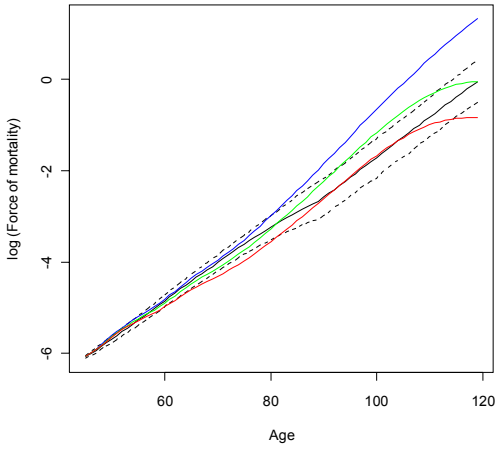


Figure 5.16 – Projected forces of mortality for England and Wales male data under the high improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

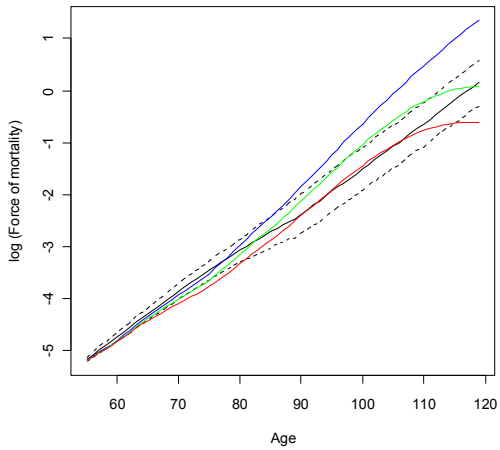
(i)



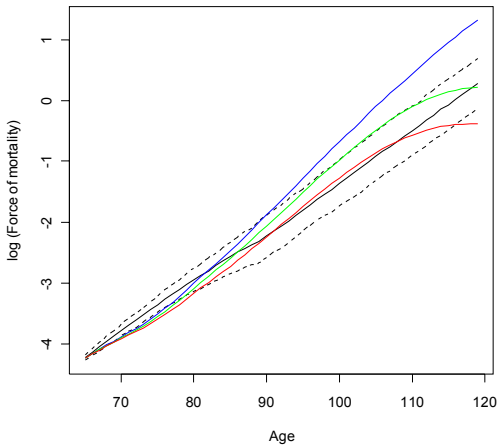
(ii)



(iii)



(iv)



(v)

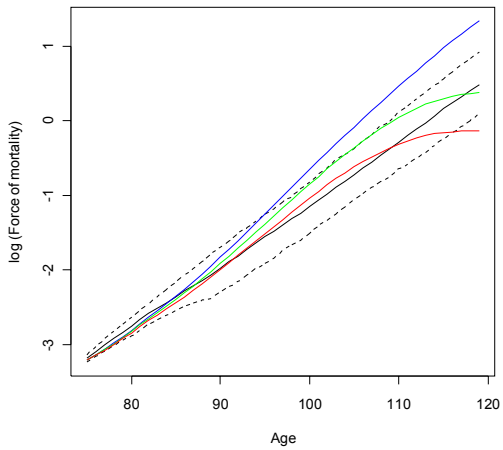
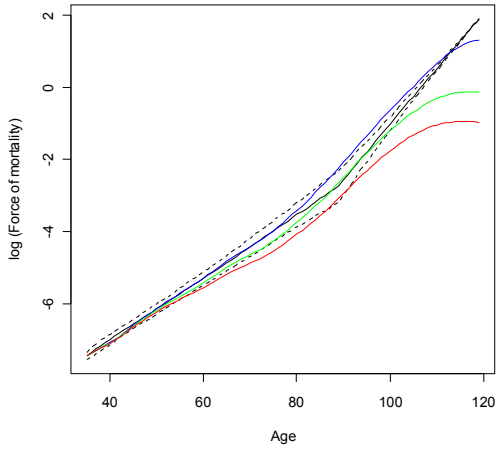
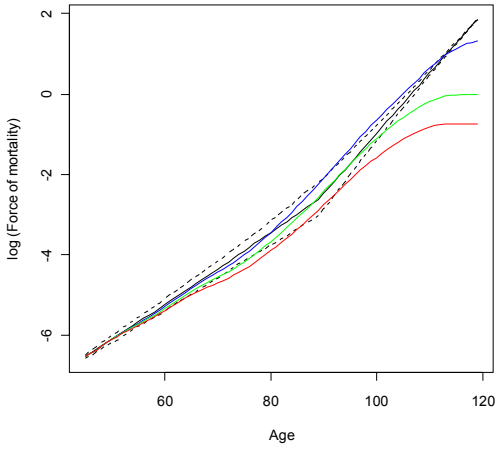


Figure 5.17 – Projected forces of mortality for England and Wales female data under the low improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

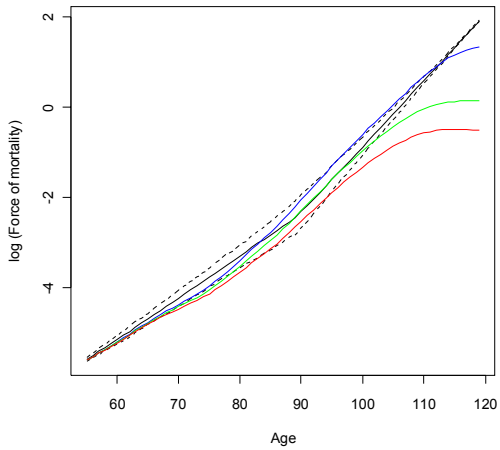
(i)



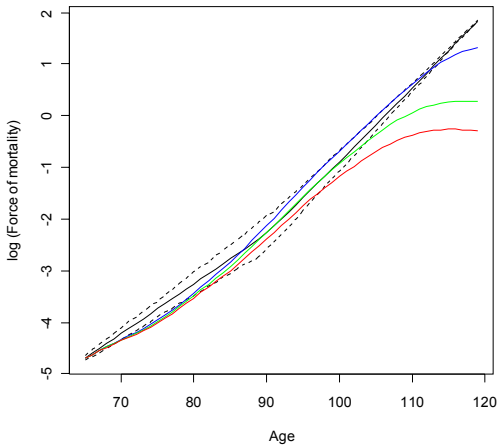
(ii)



(iii)



(iv)



(v)

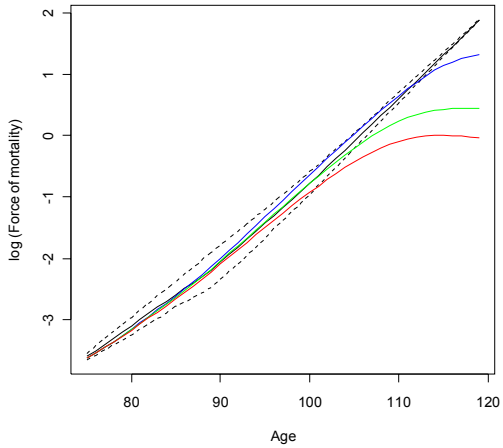
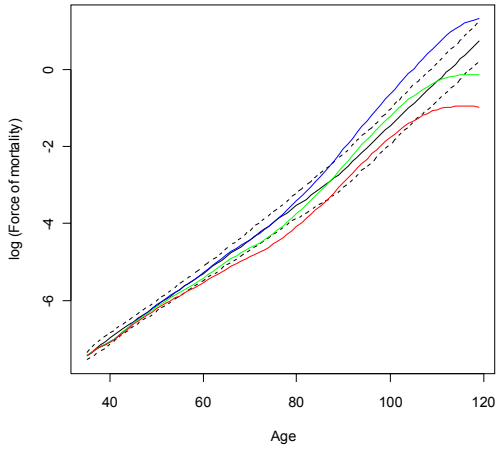
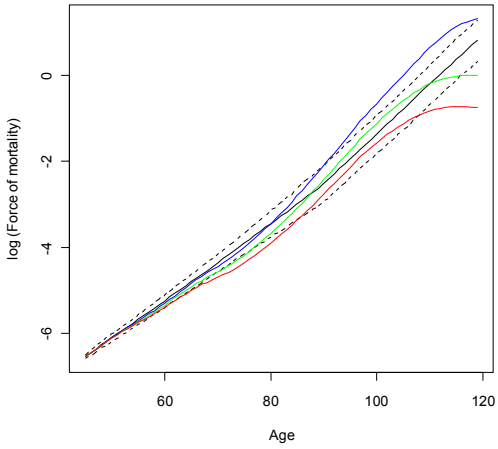


Figure 5.18 – Projected forces of mortality for England and Wales female data under the high improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

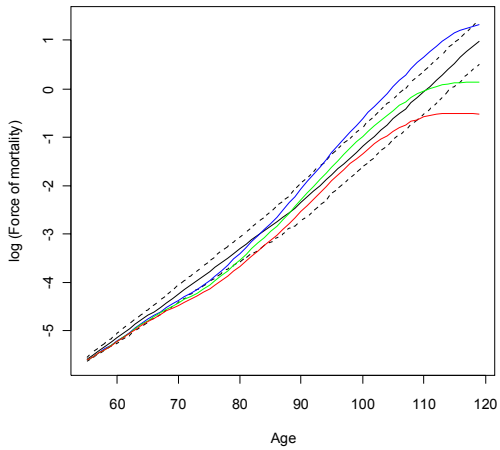
(i)



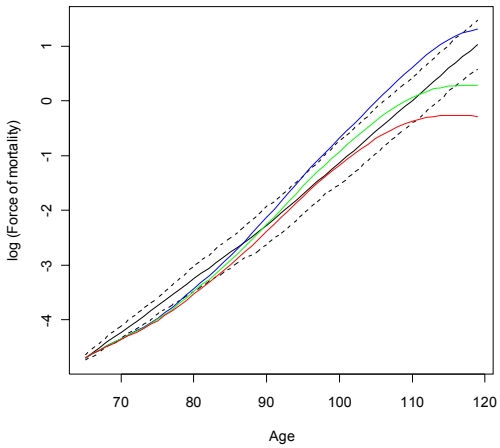
(ii)



(iii)



(iv)



(v)

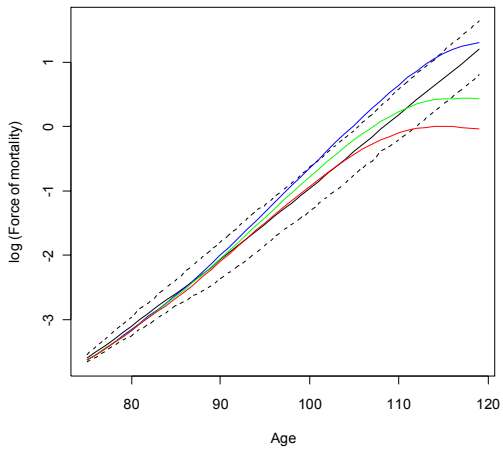
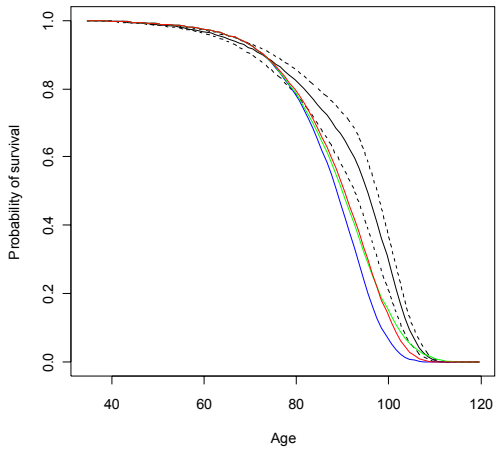
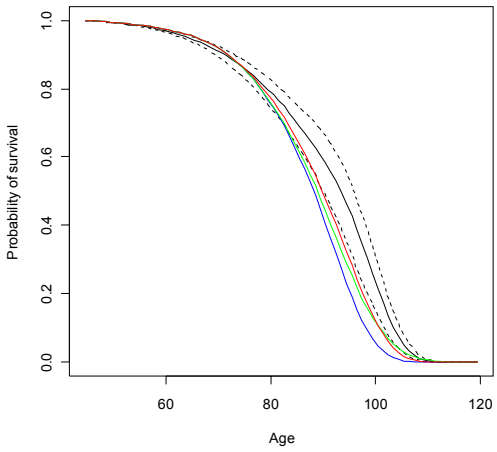


Figure 5.19 – Projected probabilities of survival for CMI data under the low improvement assumption compared with previously published projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = Medium Cohort, green solid curve = Medium Cohort subject to a 1% underpin, red solid curve = Long Cohort.

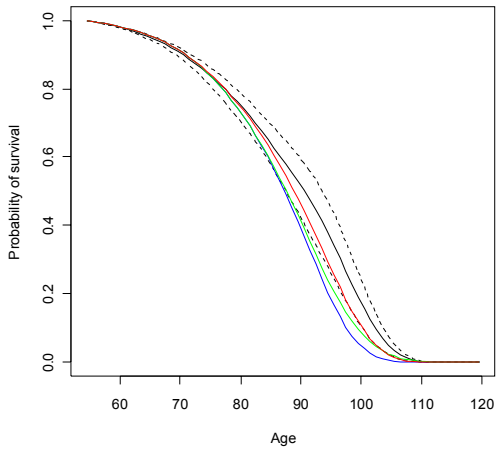
(i)



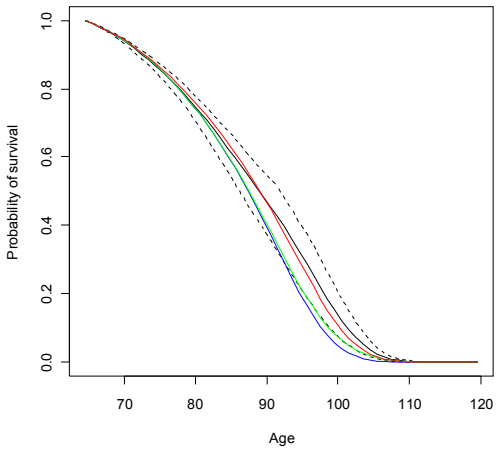
(ii)



(iii)



(iv)



(v)

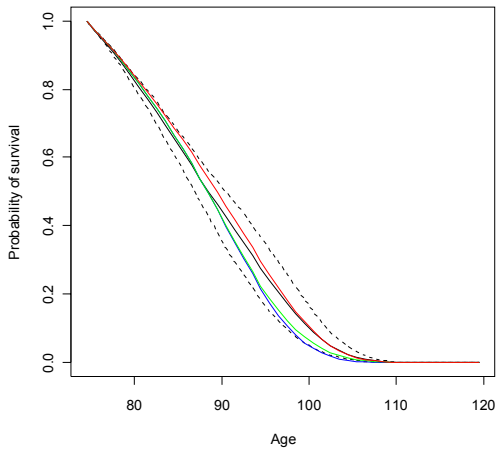
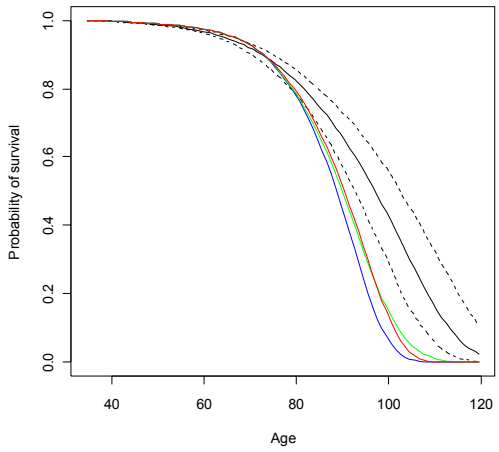
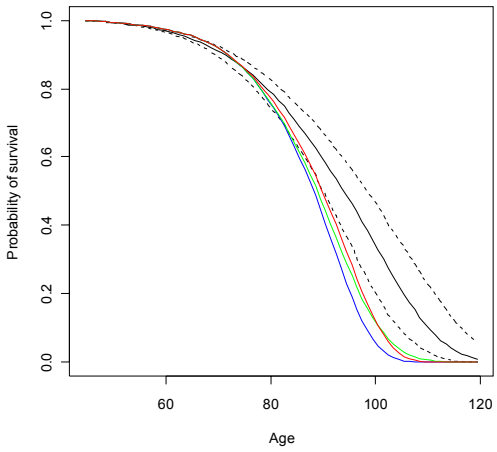


Figure 5.20 – Projected probabilities of survival for CMI data under the high improvement assumption compared with previously published projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = Medium Cohort, green solid curve = Medium Cohort subject to a 1% underpin, red solid curve = Long Cohort.

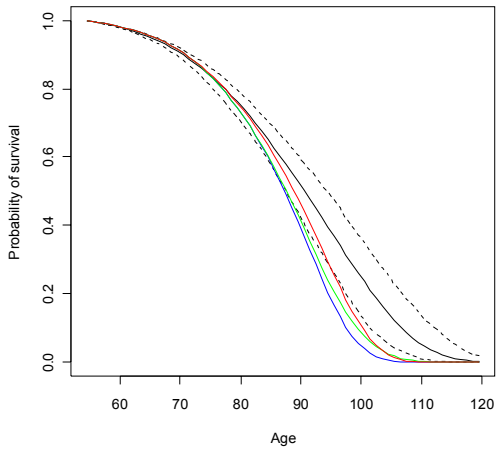
(i)



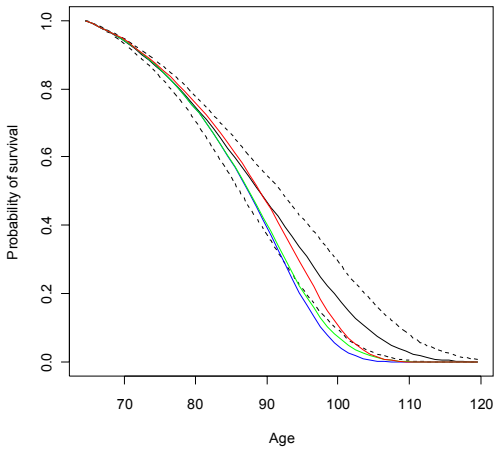
(ii)



(iii)



(iv)



(v)

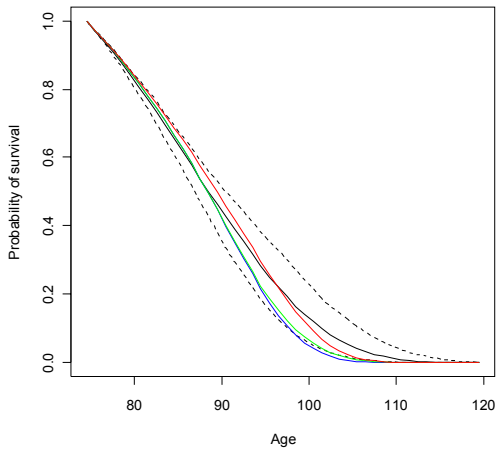
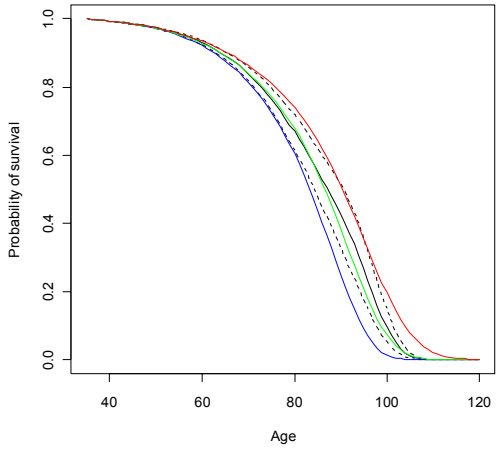
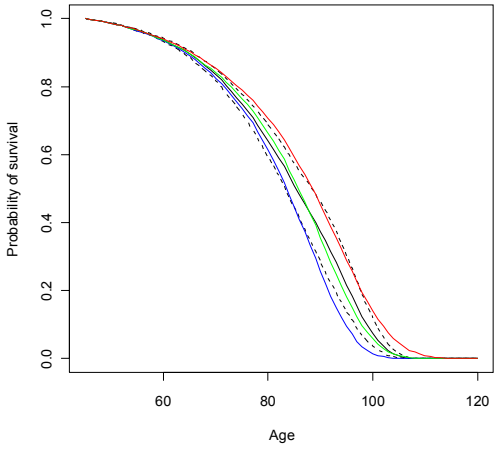


Figure 5.21 – Projected probabilities of survival for England and Wales male data under the low improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

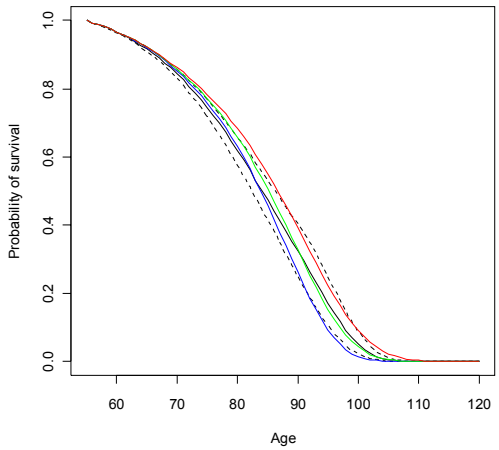
(i)



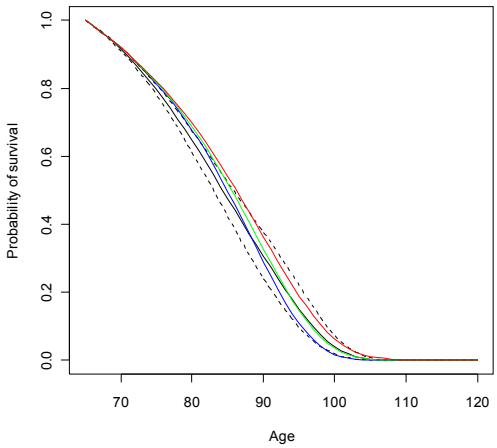
(ii)



(iii)



(iv)



(v)

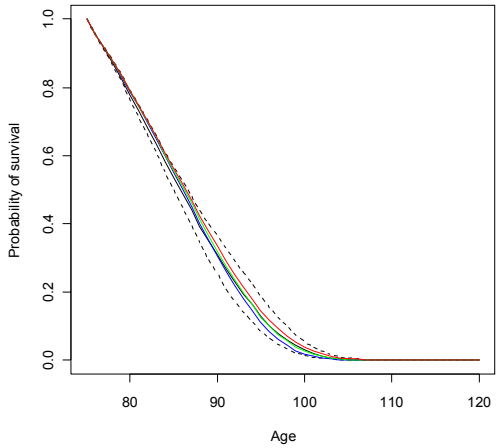
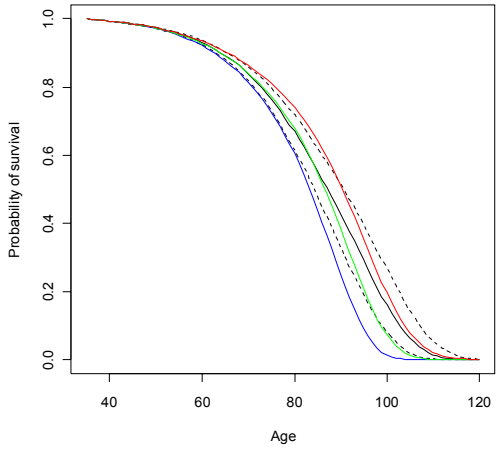
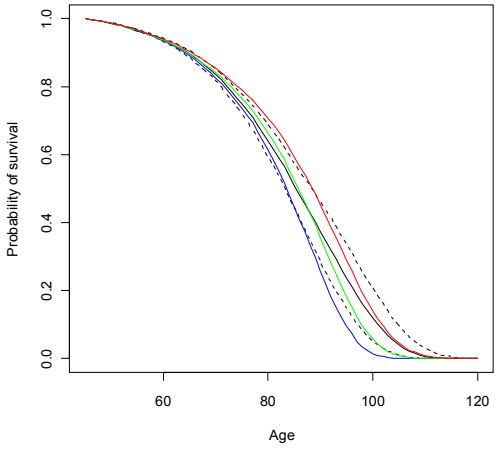


Figure 5.22 – Projected probabilities of survival for England and Wales male data under the high improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

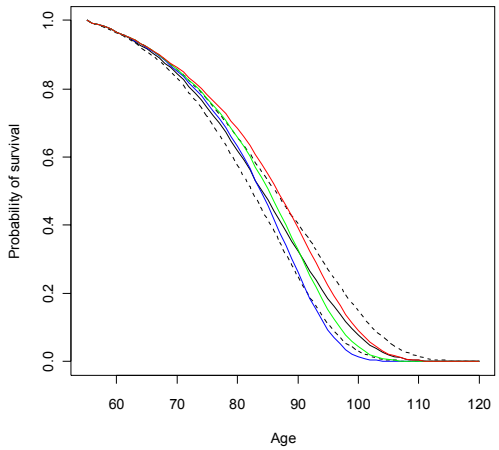
(i)



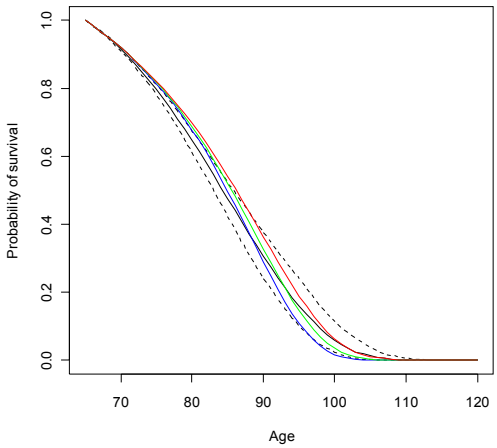
(ii)



(iii)



(iv)



(v)

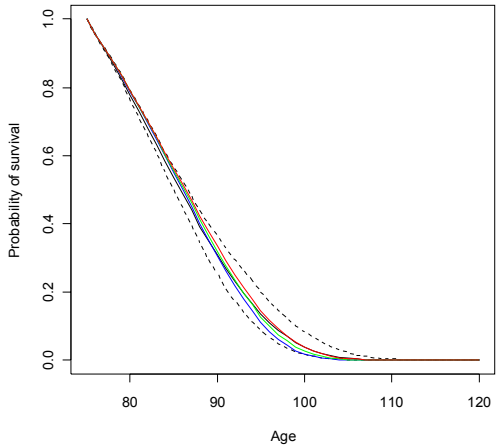
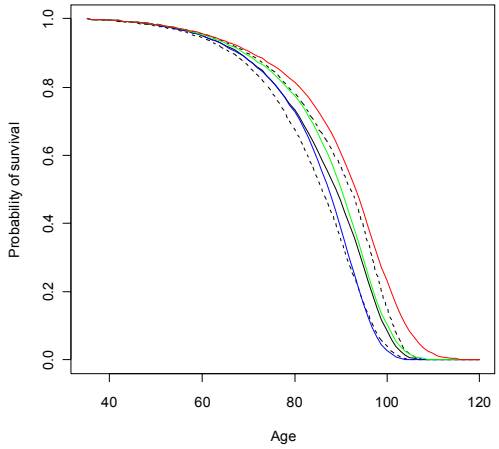
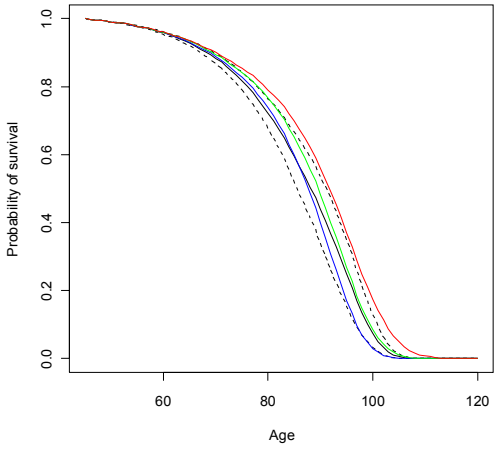


Figure 5.23 – Projected probabilities of survival for England and Wales female data under the low improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

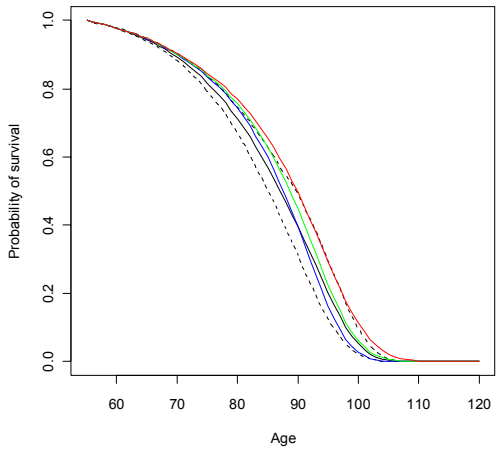
(i)



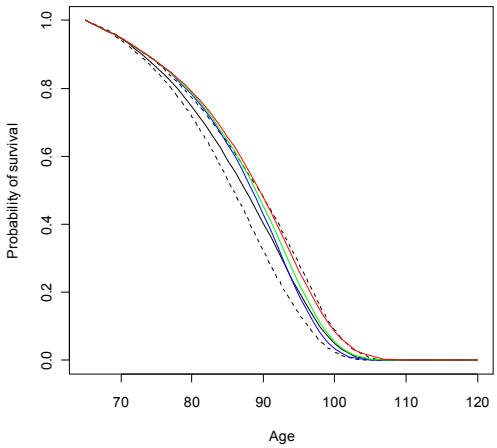
(ii)



(iii)



(iv)



(v)

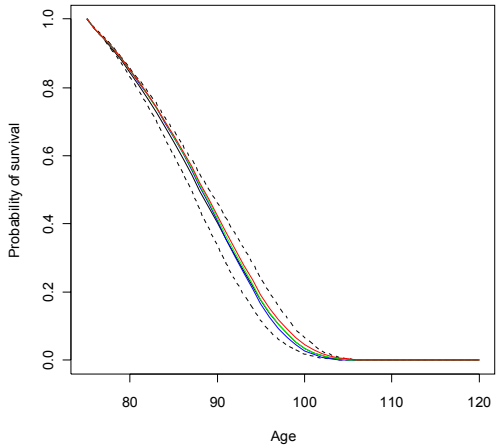
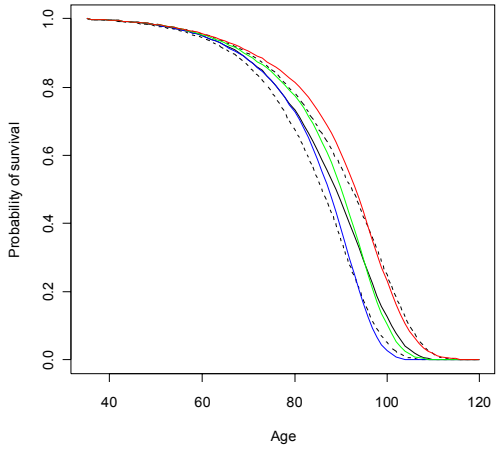
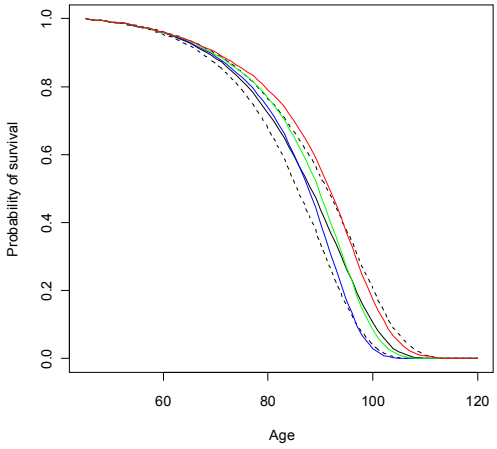


Figure 5.24 – Projected probabilities of survival for England and Wales female data under the high improvement assumption compared with GAD projection bases – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, blue solid curve = LLE, green solid curve = Principal, red solid curve = HLE.

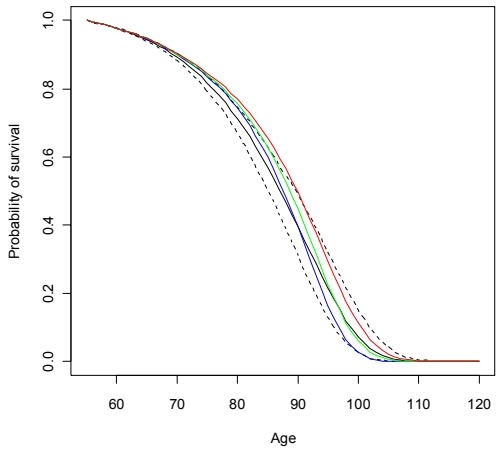
(i)



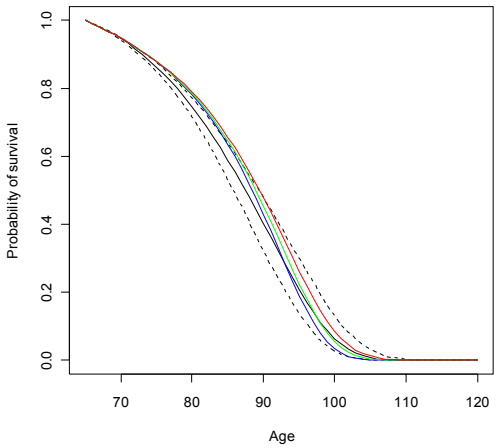
(ii)



(iii)



(iv)



(v)

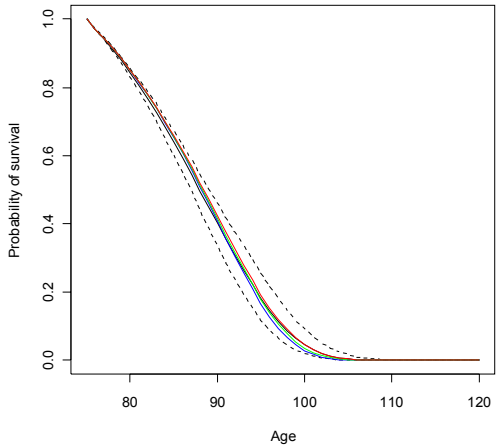
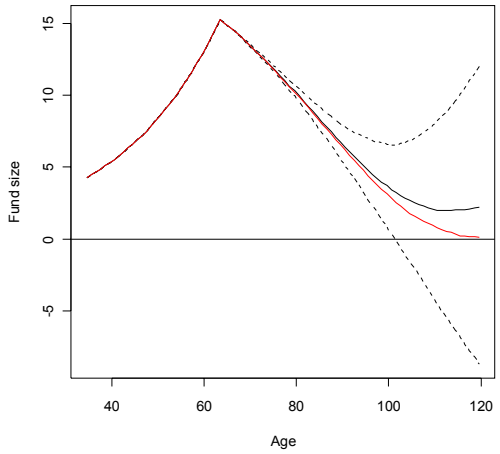
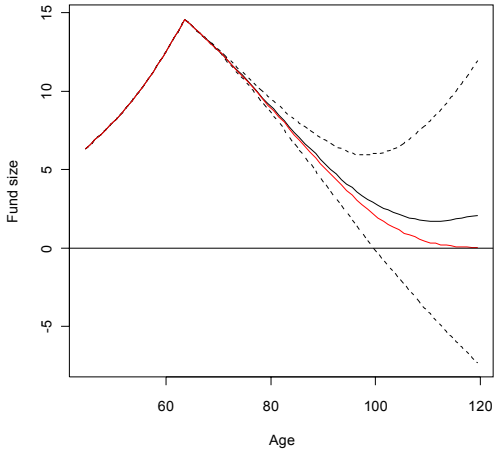


Figure 5.25 – Projected future sizes of a fund set up in 2005 equal to the mean annuity value plus ICA capital for CMI data – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, red solid curve = 1 - percentile specified in Table 5.18.

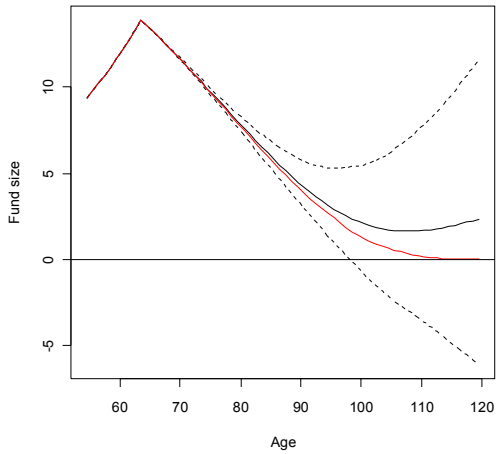
(i)



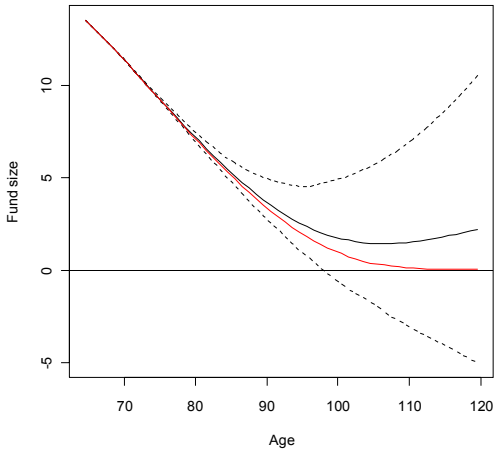
(ii)



(iii)



(iv)



(v)

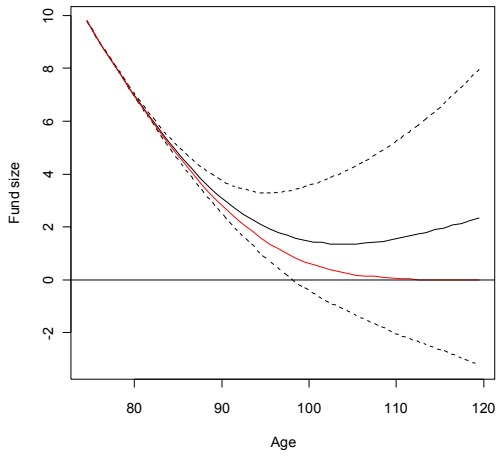
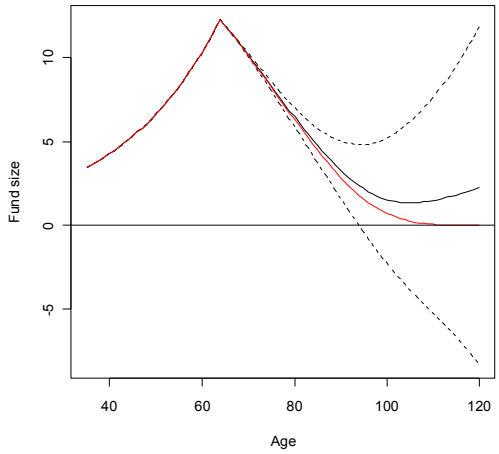
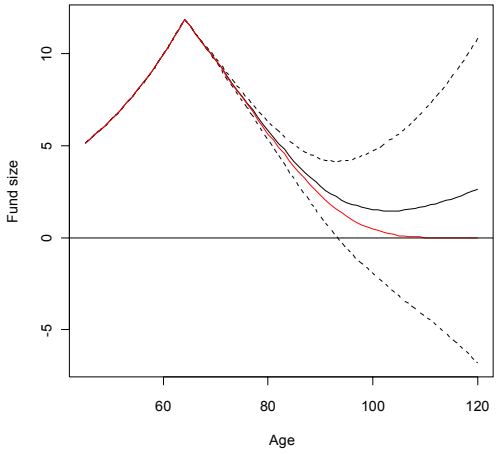


Figure 5.26 – Projected future sizes of a fund set up in 2005 equal to the mean annuity value plus ICA capital for England and Wales male data – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, red solid curve = 1 - percentile specified in Table 5.18.

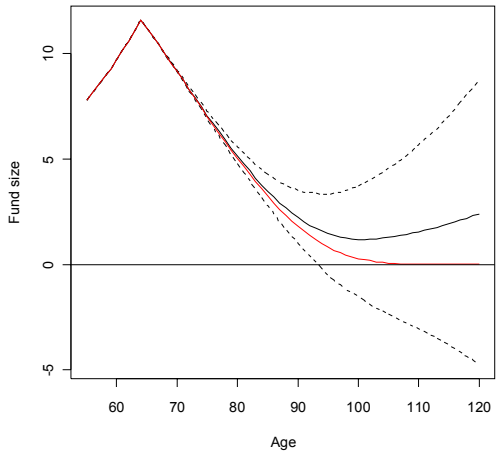
(i)



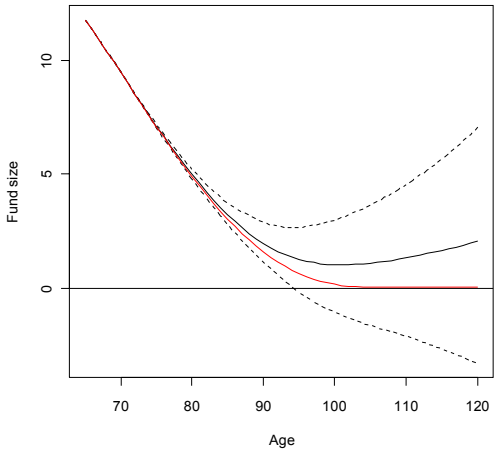
(ii)



(iii)



(iv)



(v)

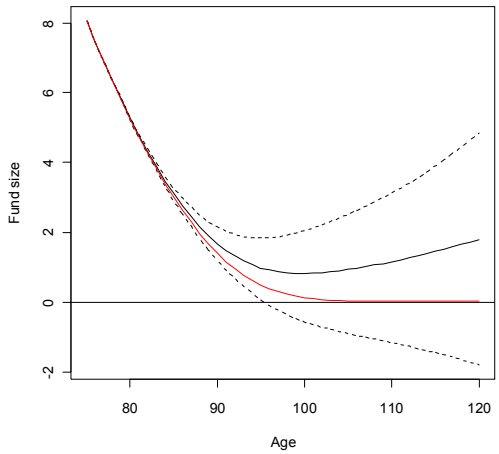
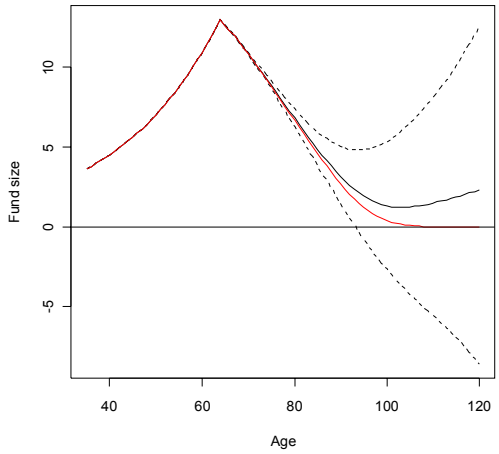
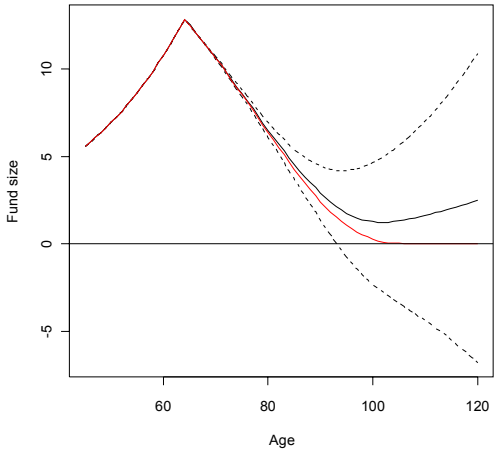


Figure 5.27 – Projected future sizes of a fund set up in 2005 equal to the mean annuity value plus ICA capital for England and Wales female data – age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75. Black solid curve = median, black dashed curves = 2.5th and 97.5th percentiles, red solid curve = 1 - percentile specified in Table 5.18.

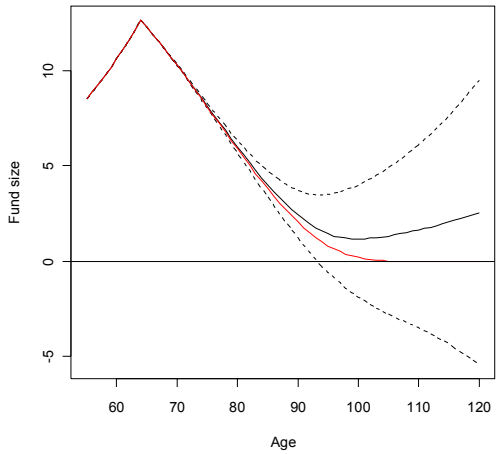
(i)



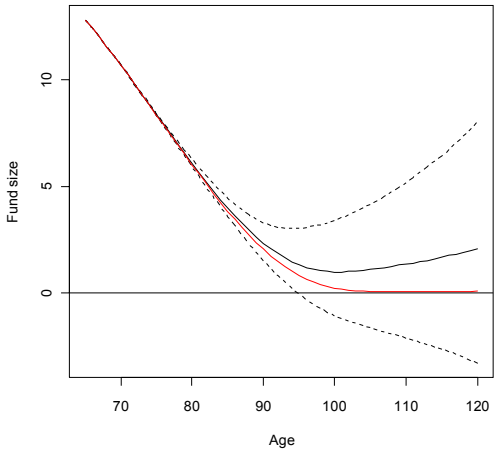
(ii)



(iii)



(iv)



(v)

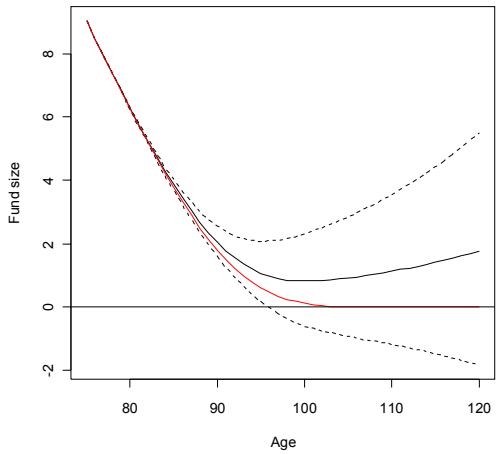
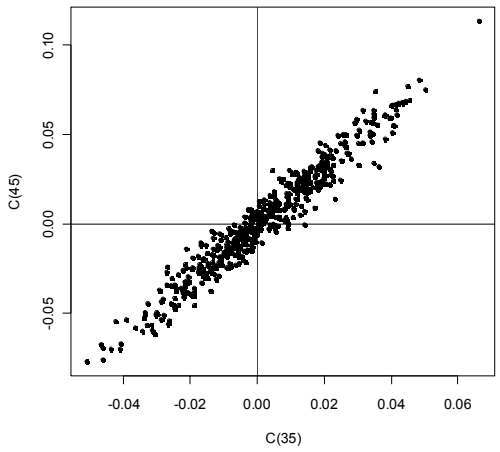
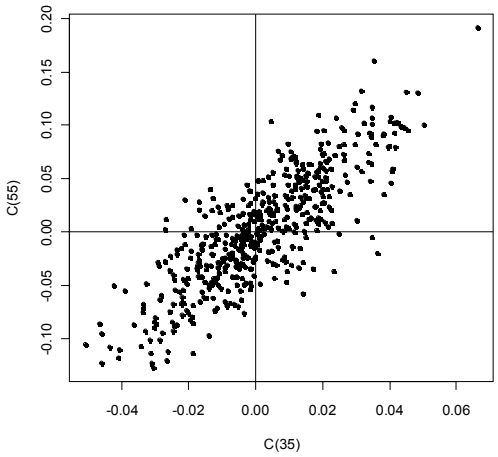


Figure 5.28 – Scatter diagrams to illustrate the correlation between the values of $C(x)$ at different ages x in 2005 for CMI data **–** (i) 35 *v* 45, (ii) 35 *v* 55, (iii) 35 *v* 65, (iv) 35 *v* 75

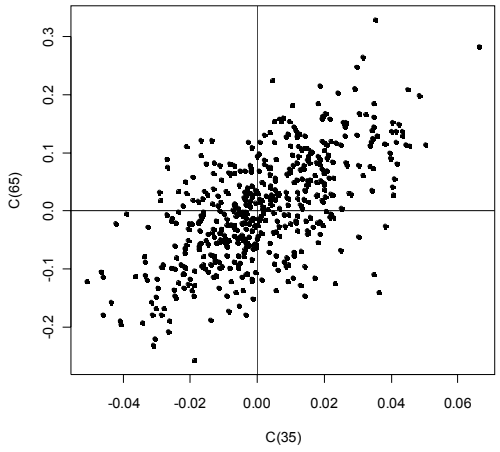
(i)



(ii)



(iii)



(iv)

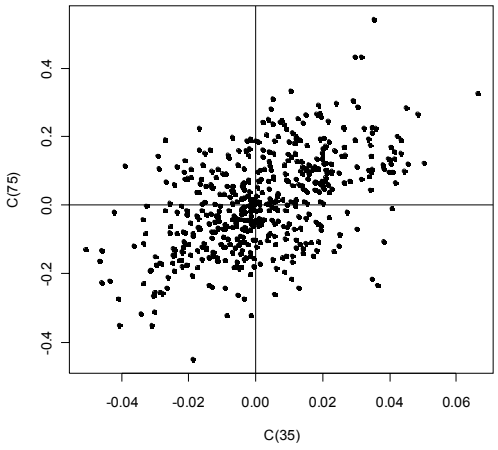
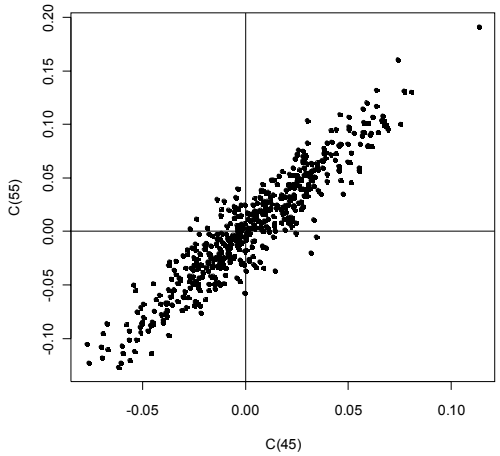
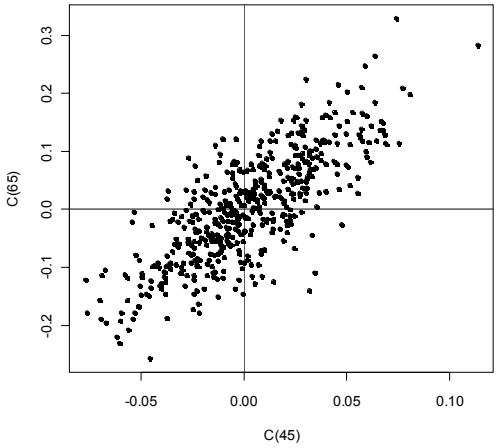


Figure 5.28 (continued) **–** Scatter diagrams to illustrate the correlation between the values of $C(x)$ at different ages x in 2005 for CMI data **–** (v) 45 *v* 55, (vi) 45 *v* 65, (vii) 45 *v* 75, (viii) 55 *v* 65, (ix) 55 *v* 75, (x) 65 *v* 75

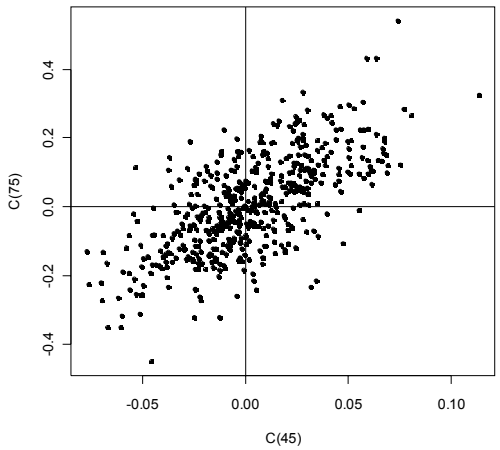
(v)



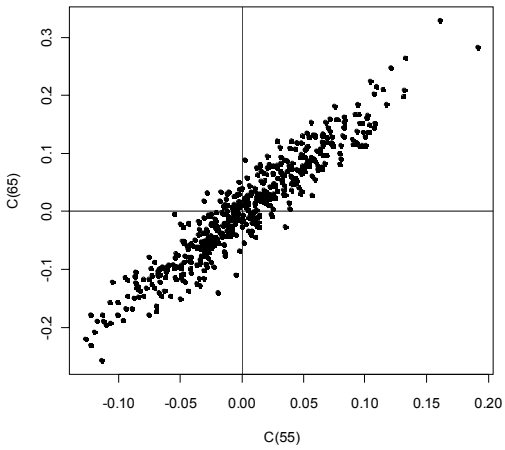
(vi)



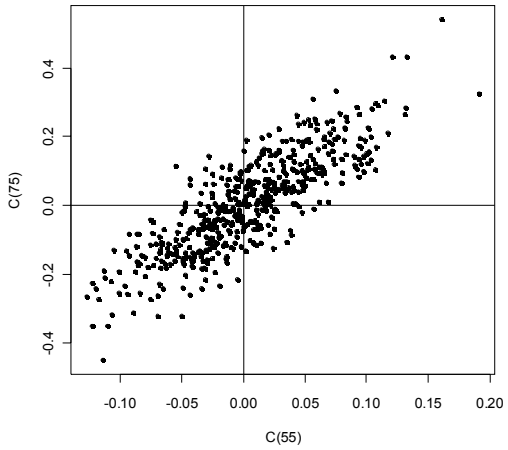
(vii)



(viii)



(ix)



(x)

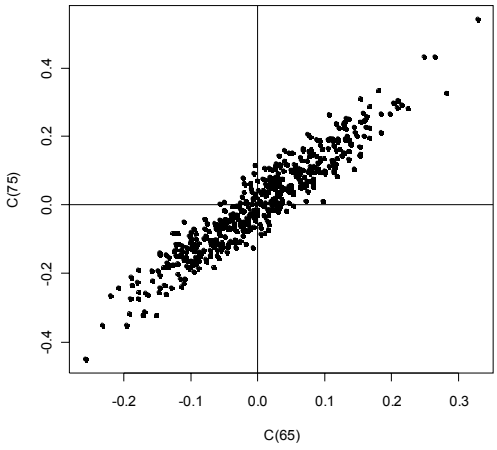
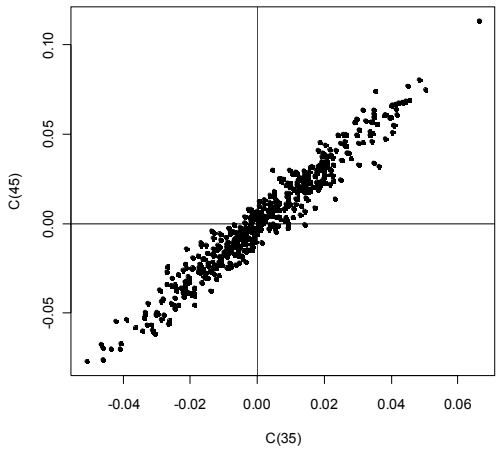
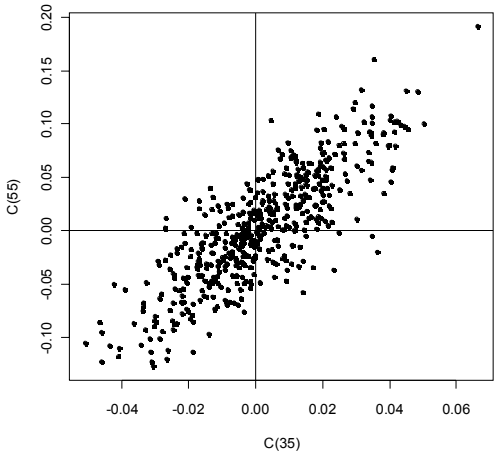


Figure 5.29 – Scatter diagrams to illustrate the correlation between the values of $C(x)$ at different ages x in 2005 for England and Wales male data **–** (i) 35 *v* 45, (ii) 35 *v* 55, (iii) 35 *v* 65, (iv) 35 *v* 75

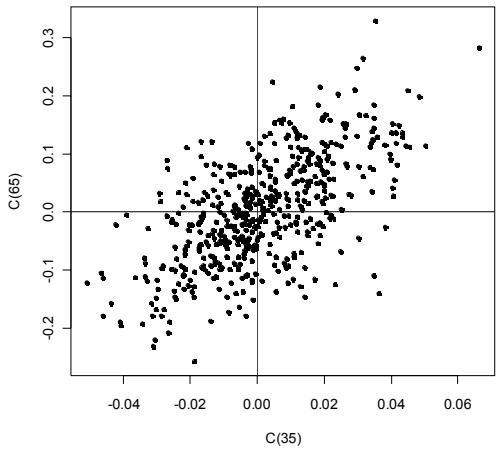
(i)



(ii)



(iii)



(iv)

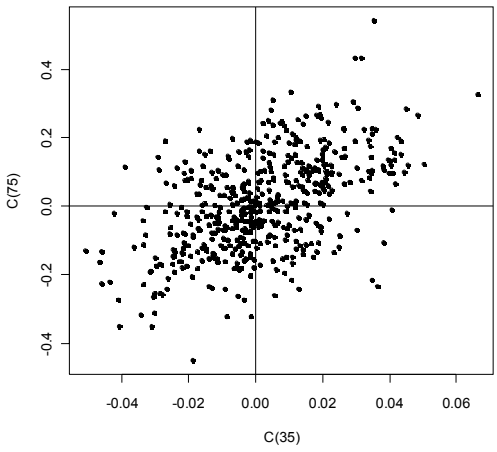
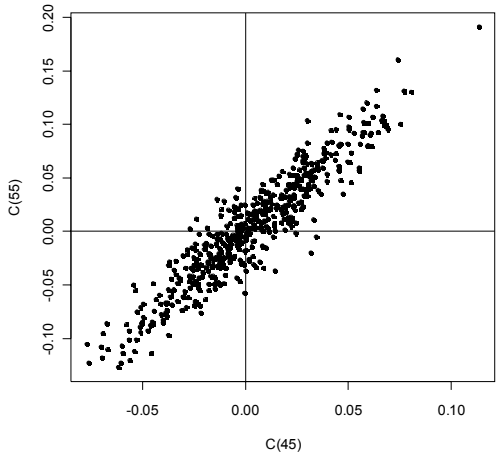
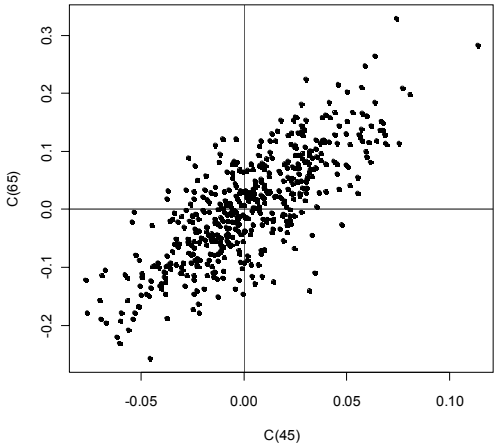


Figure 5.29 (continued) **–** Scatter diagrams to illustrate the correlation between the values of $C(x)$ at different ages x in 2005 for England and Wales male data **–** (v) 45 *v* 55, (vi) 45 *v* 65, (vii) 45 *v* 75, (viii) 55 *v* 65, (ix) 55 *v* 75, (x) 65 *v* 75

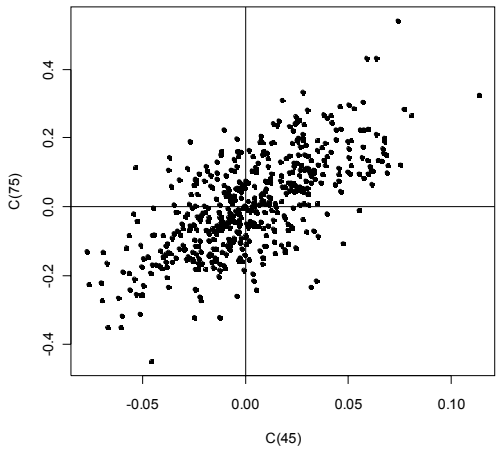
(v)



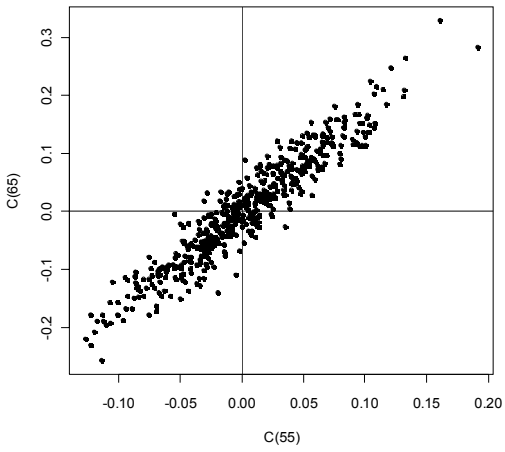
(vi)



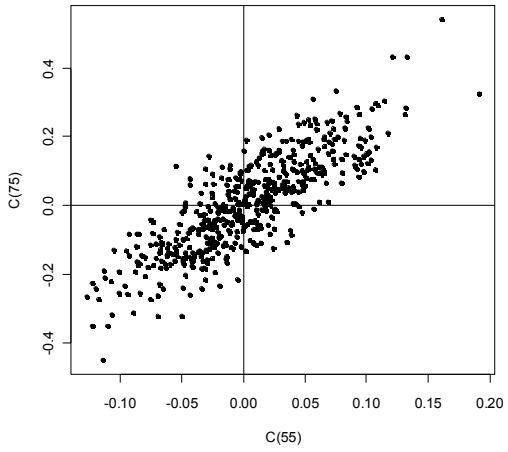
(vii)



(viii)



(ix)



(x)

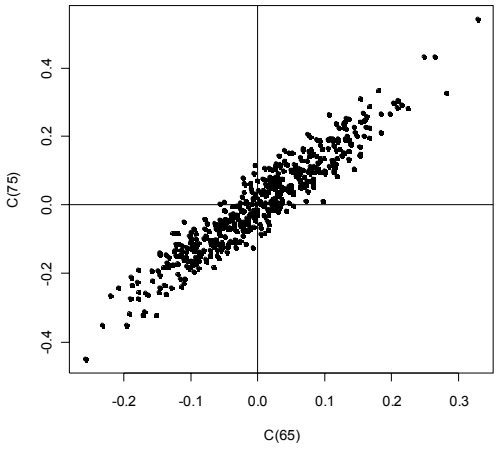
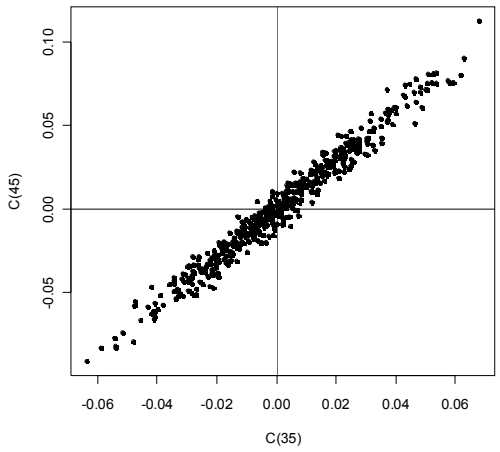
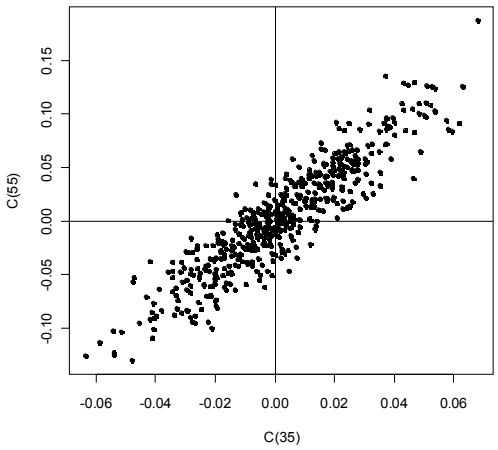


Figure 5.30 – Scatter diagrams to illustrate the correlation between the values of $C(x)$ at different ages x in 2005 for England and Wales female data **–** (i) 35 *v* 45, (ii) 35 *v* 55, (iii) 35 *v* 65, (iv) 35 *v* 75

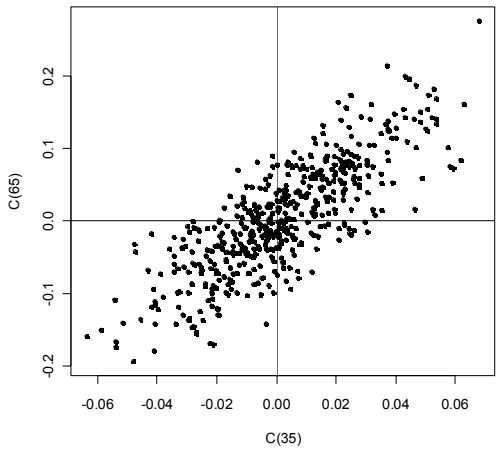
(i)



(ii)



(iii)



(iv)

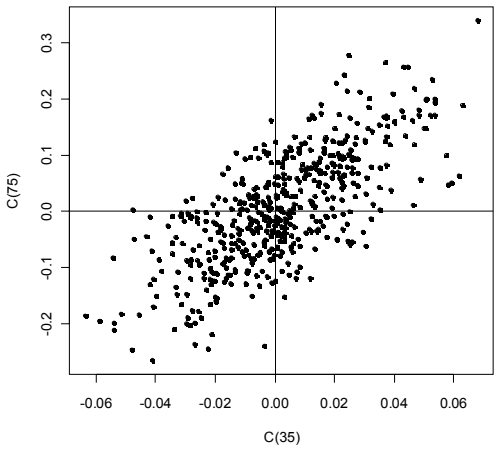
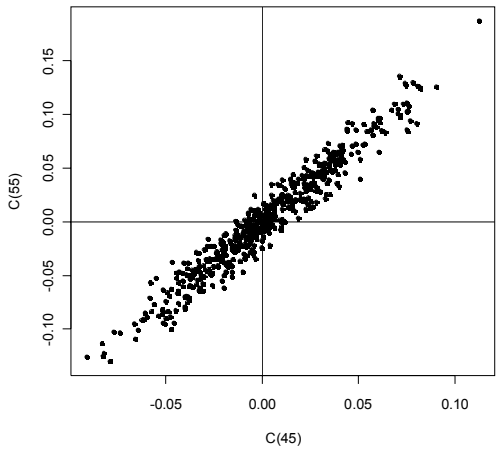
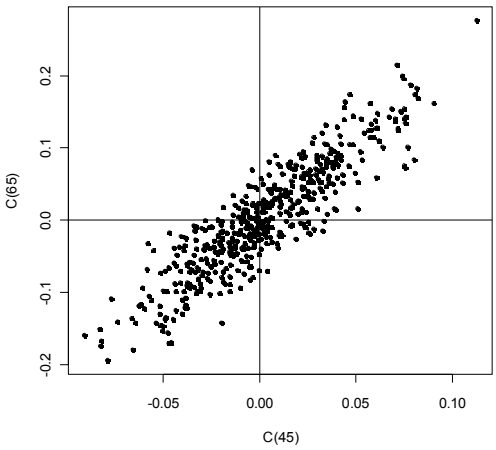


Figure 5.30 (continued) **–** Scatter diagrams to illustrate the correlation between the values of $C(x)$ at different ages x in 2005 for England and Wales female data **–** (v) 45 *v* 55, (vi) 45 *v* 65, (vii) 45 *v* 75, (viii) 55 *v* 65, (ix) 55 *v* 75, (x) 65 *v* 75

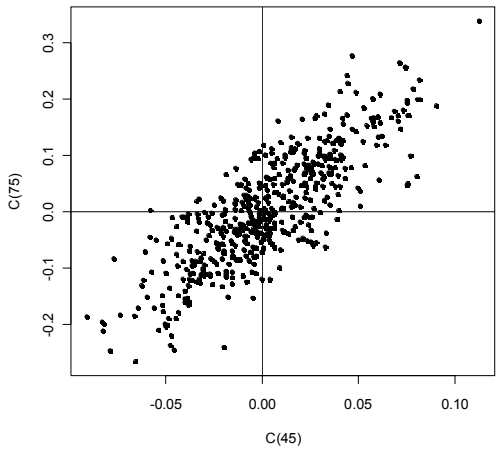
(v)



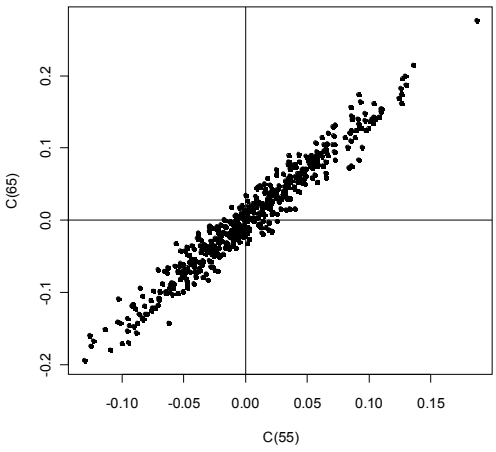
(vi)



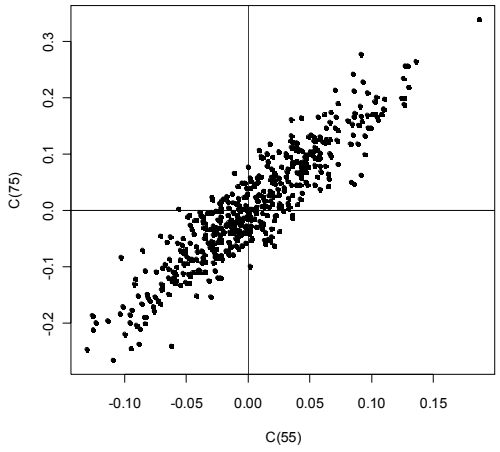
(vii)



(viii)



(ix)



(x)

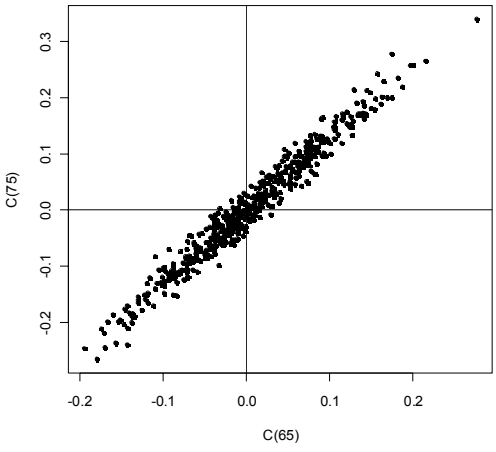
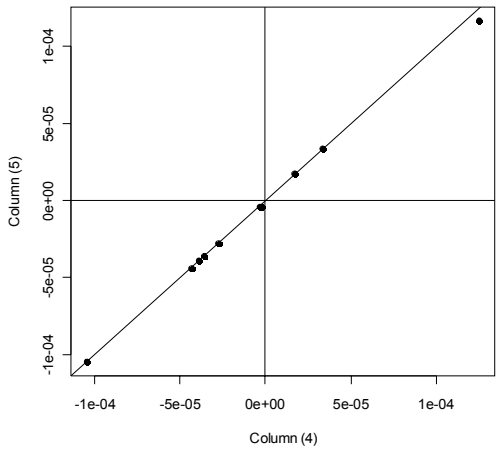
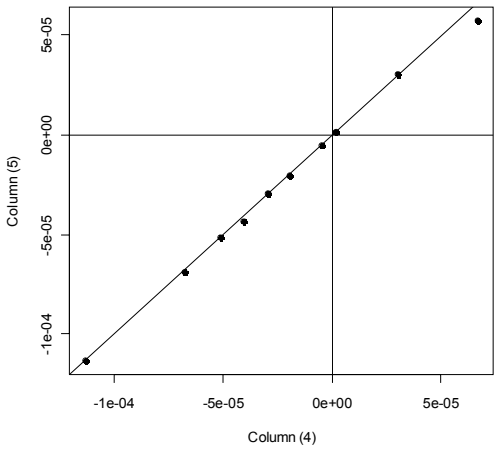


Figure 5.31 – Illustration of the accuracy of Approximation 1 for CMI data **–** columns (4) and (5) are as in Table 5.24 **–** age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75

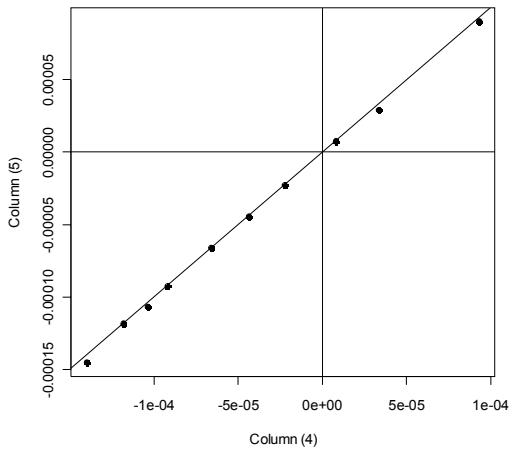
(i)



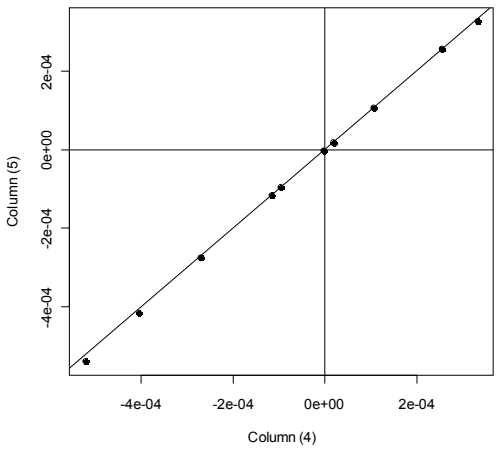
(ii)



(iii)



(iv)



(v)

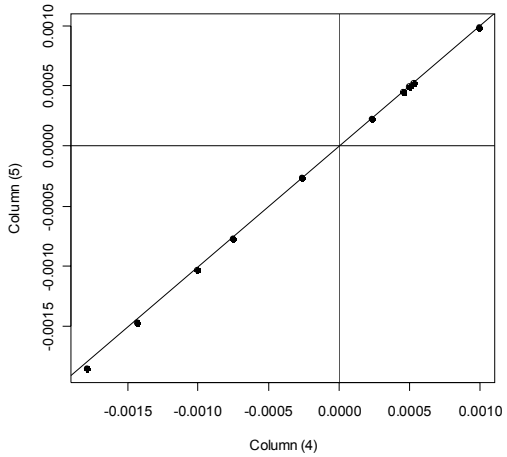
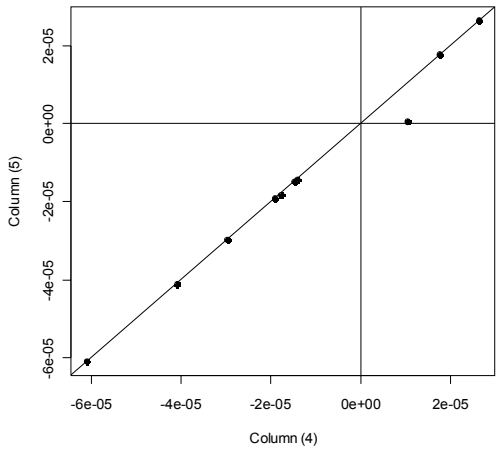
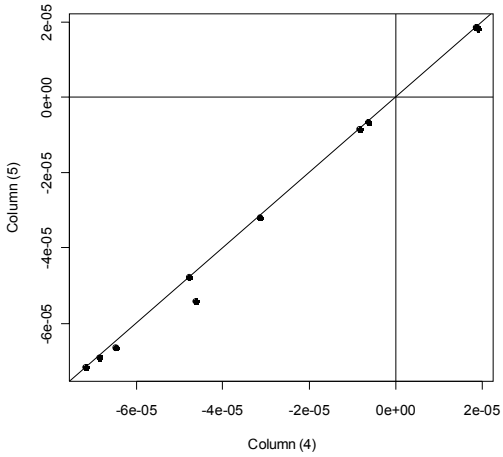


Figure 5.32 – Illustration of the accuracy of Approximation 1 for England and Wales male data **–** columns (4) and (5) are as in Table 5.25 **–** age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75

(i)



(ii)



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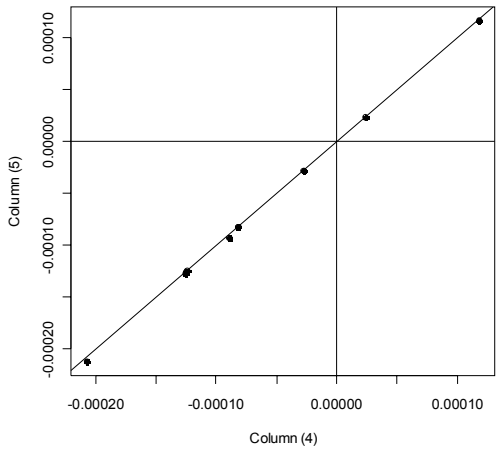
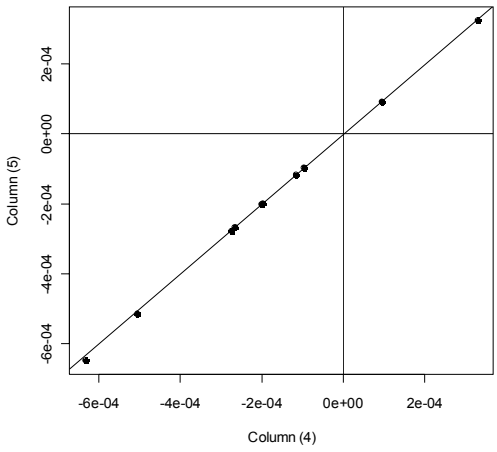


(iv)



(v)

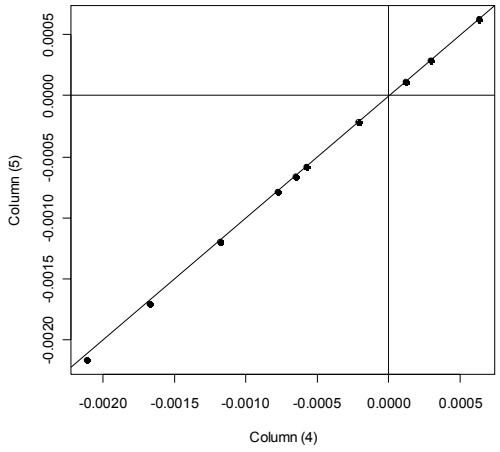
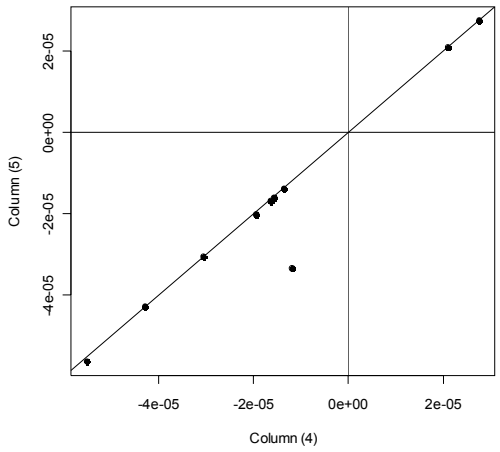
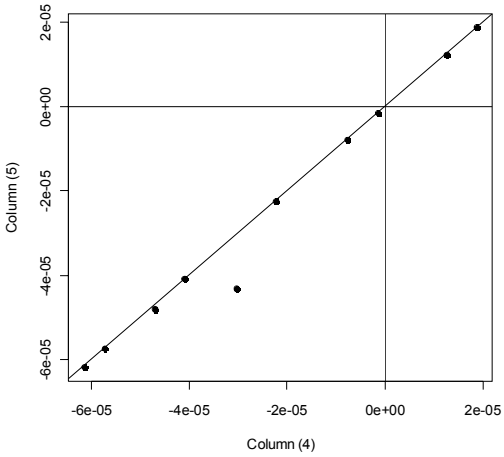


Figure 5.33 – Illustration of the accuracy of Approximation 1 for England and Wales female data **–** columns (4) and (5) are as in Table 5.26 **–** age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75

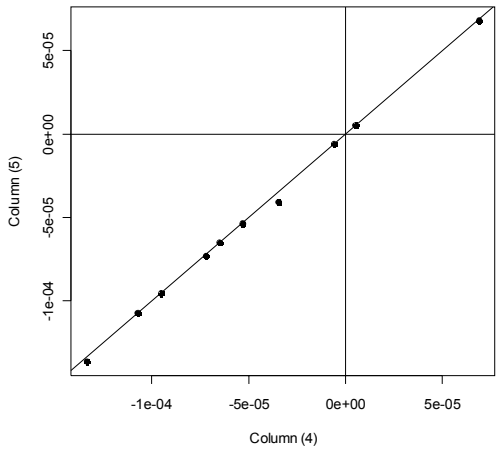
(i)



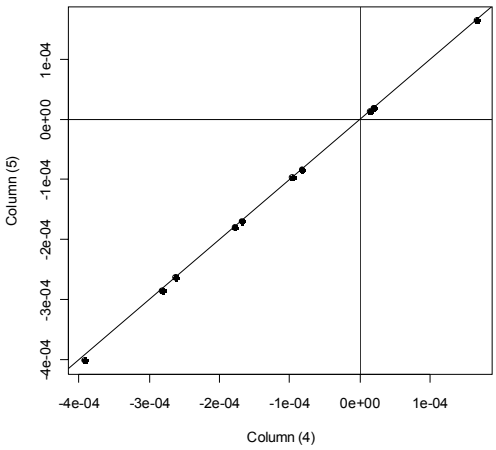
(ii)



(iii)



(iv)



(v)

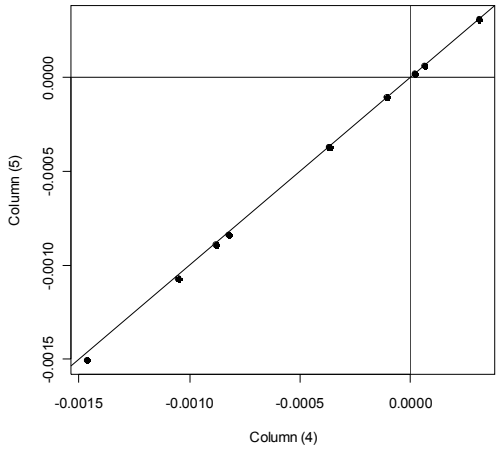
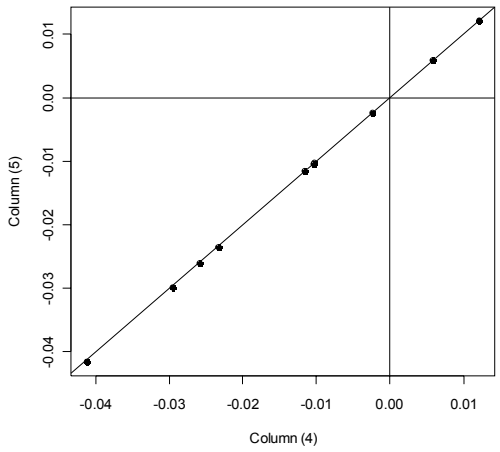
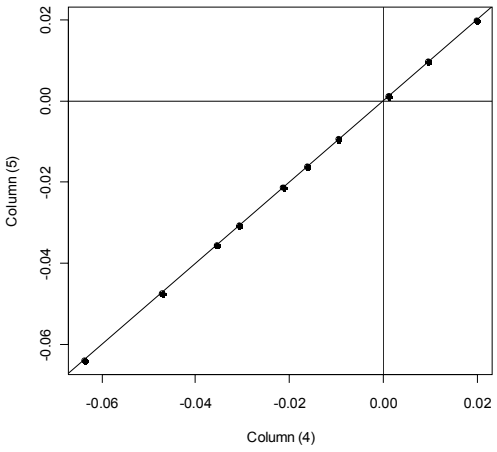


Figure 5.34 – Illustration of the accuracy of Approximation 2 for CMI data **–** columns (4) and (5) are as in Table 5.27 **–** age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75

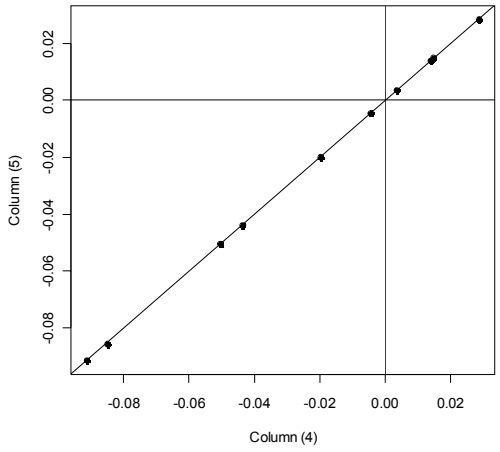
(i)



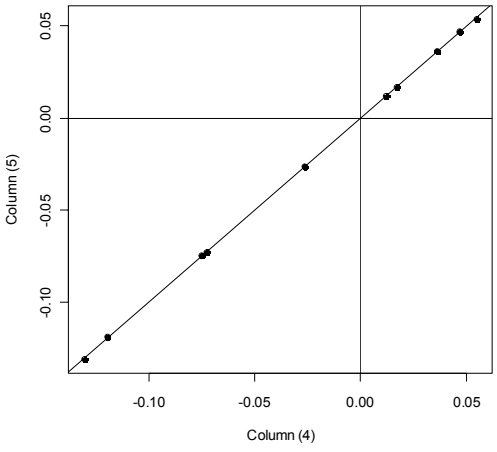
(ii)



(iii)



(iv)



(v)

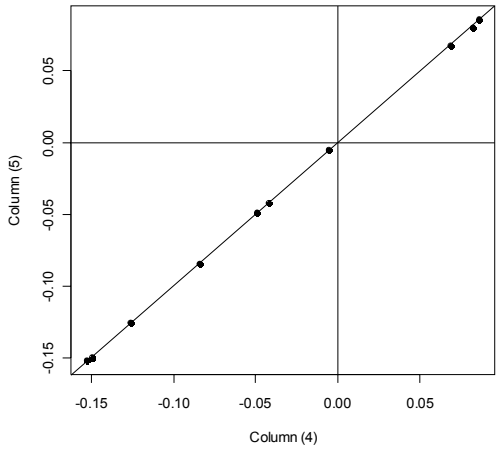
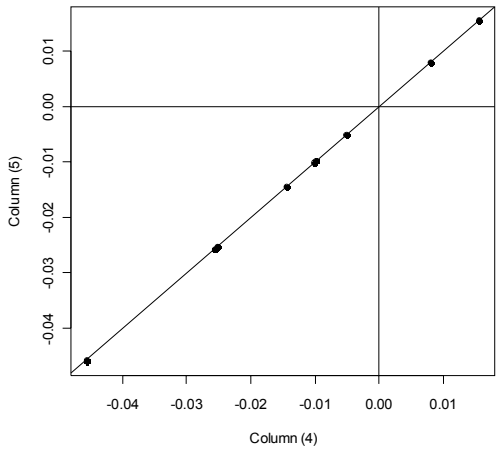
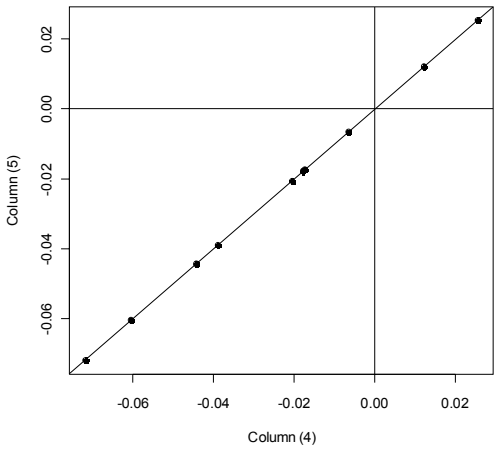


Figure 5.35 – Illustration of the accuracy of Approximation 2 for England and Wales male data **–** columns (4) and (5) are as in Table 5.28 **–** age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75

(i)



(ii)



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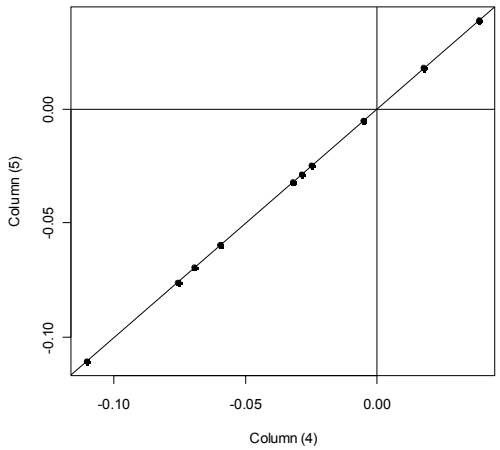
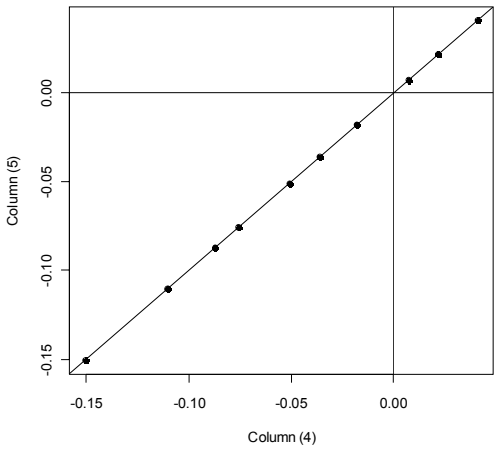


(iv)



(v)

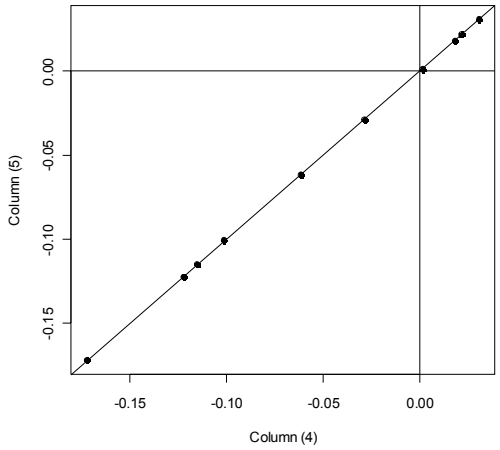
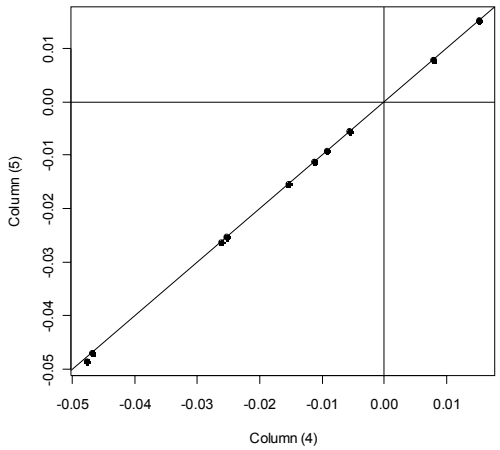
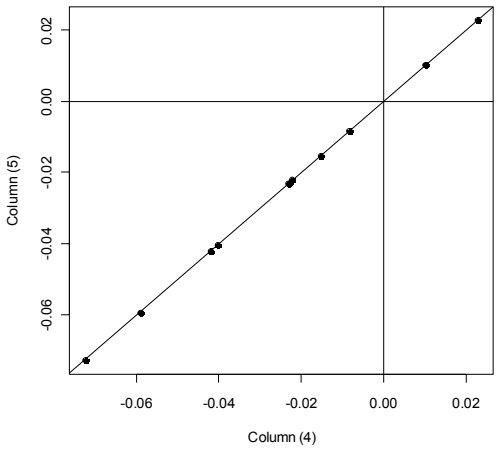


Figure 5.36 – Illustration of the accuracy of Approximation 2 for England and Wales female data **–** columns (4) and (5) are as in Table 5.29 **–** age in 2005 (i) 35, (ii) 45, (iii) 55, (iv) 65, (v) 75

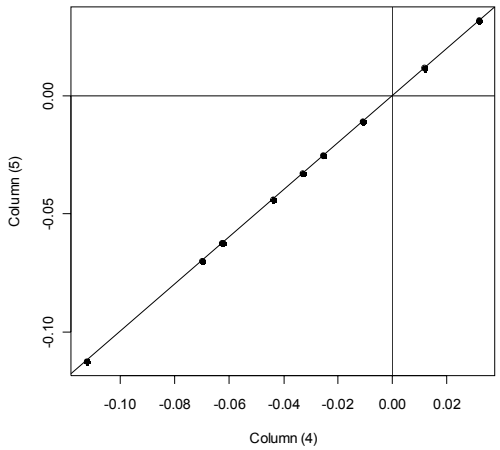
(i)



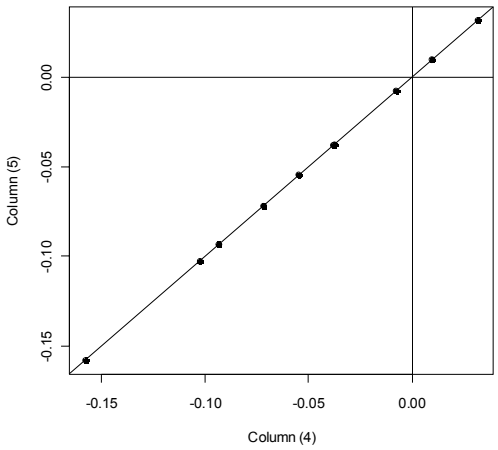
(ii)



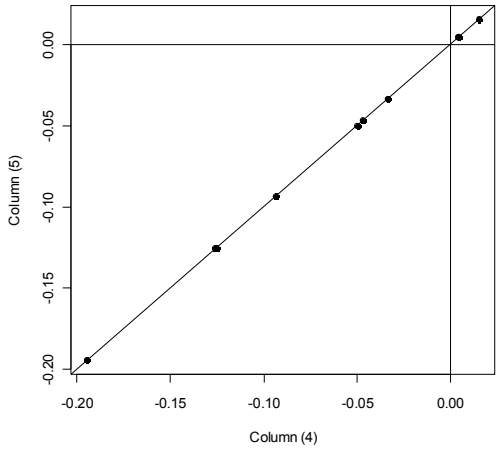
(iii)



(iv)



(v)



6: Conclusions

In Chapter 2, after fitting a number of Gompertz-Makeham models with time-dependent coefficients, the kappa parameters, to CMI data, England and Wales male data and England and Wales female data, we concluded that the GM(1,3) model gave a satisfactory fit to all three data sets and that satisfactory explanations could be given for the shapes of the resulting parameter graphs. However, some of the explanations depended upon cohort effects, and there were also cohort effects visible in the residuals, particularly for the England and Wales data sets. As a result, it was decided to introduce a further parameter γ_c depending on year of birth c into the model, applied to the force of mortality under the GM(1,3) model as a multiplicative factor.

Chapter 3 estimated γ_c, for each year of birth c, as the ratio of the actual number of deaths for year of birth c to the expected number of deaths under the model not including gamma parameters. For England and Wales data, the improvement in fit achieved by introducing the gamma parameters was found to be very significant. For CMI data, the improvement was more marginal – it appeared that this was because most of the cohort effects had already been captured implicitly via the shapes of the kappa parameter graphs and also because the smaller volume of the CMI data made any remaining cohort effects less statistically significant. Nevertheless, the similarities between the shapes of the gamma parameter graphs obtained for CMI data and for England and Wales data provided justification for introducing the gamma parameters for CMI data as well. Further improvement in fit to all three data sets, in a pure maximum likelihood sense, could have been achieved by re-estimating the kappa parameters following the estimation of the gamma parameters, but this was not implemented because it was found that the gamma parameters then reflected effects that were clearly not genuine cohort effects.

Chapter 4 fitted univariate time series models to the estimates of both the kappa and the gamma parameters. For the $\kappa^{(3)}$ parameters, representing the general level of mortality in each calendar year, there was found to be a significant downward trend, which was greatest for CMI data and least for England and Wales female data, and an ARIMA(0,1,1) model was used. For the other parameters, an AR(1) model was used. In the case of the $\kappa^{(0)}$ parameters, it was considered appropriate to use standard techniques to estimate the parameters of this AR(1) model. In the case of the $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters, consideration of the impact of cohort effects was used to suggest a long-term mean value which was not necessarily similar to what would have been obtained using standard techniques. As the procedure we used to estimate the γ parameters was such that their logarithms could be away from zero only over a limited period, an AR(1) model with a zero mean value was fitted to the logarithms. The reason for fitting a time series model to the logarithms, rather than to the γ parameters themselves, was to eliminate the possibility of negative γ parameters.

In Chapter 5, we generated 500 scenarios for future values of the parameters for each data set, allowing for the observed correlations between the different kappa series. These scenarios were then used to calculate distributions of deferred and immediate annuity values for various values of the age in 2005 (the latest year in the data), using a deterministic interest rate. The annuity values were highest for CMI data and lowest for England and Wales male data, confirming that the mortality differential between assured lives and the general population is greater than that between males and females. It was necessary to extrapolate the 2005 mortality curve above the highest age in the data, and two different methods of doing this were considered. It was also necessary to make an assumption about future mortality improvements at ages above the highest age in the data, and two different assumptions were again considered to give an indication of the range of potential outcomes. Some comparisons were presented of the annuity values against previously published projections, starting from the 2005 mortality curves we fitted. In most scenarios, the model of this thesis gave higher annuity values than the previously published projections for lives aged 65 and 75 in 2005 in CMI data, but lower values for England and Wales data for both males and females. The model of this thesis generally gave higher future mortality improvements for each generation than the previously published projections at ages below the highest age of the data, as evidenced by the trend for the annuity values from the model of this thesis to

increase relative to the values from the previously published projections as the age in 2005 decreased below 65.

In Chapter 5 we also applied the model to ICA capital calculations for deferred and immediate annuities. The ICA capital values we arrived at were significantly lower than those given by the 'rule of thumb' of taking the 95th percentile of the distribution of the annuity values over the full outstanding lifetime of the annuitants, particularly for groups of annuitants with a long future life expectancy. This possibly justifies the use of a lower percentile than the 95th percentile, but we listed three factors an insurance company should consider before deciding to use a lower percentile. We also calculated diversification benefits in terms of ICA capital for an insurer with a portfolio of annuitants of different ages rather than a single age, and we found that the benefits were most significant for CMI data and least significant for England and Wales male data.

A number of possible topics for further research can be identified:

- Further investigation into the cause of the 1919-20 discontinuity in the gamma parameters would be beneficial. We followed Renshaw and Haberman (2006) in attributing it to the 1919 influenza epidemic but it is not clear that this would have led to reduced mortality for lives born in 1919 and to increased mortality for lives born in 1920, rather than affecting mortality in some other way.

- It would be valuable to develop mortality models which place cohort effects on an equal footing with period effects and which therefore provide an unbiased assessment of whether particular features of the data are consequences of period or cohort effects. Once such a model has been developed, the problems referred to in Chapter 3 with estimating both the period and cohort parameters in a single iterative procedure can be expected to be resolved. In developing such a model, it would be necessary to avoid the convergence and robustness issues that have been identified with the Renshaw and Haberman (2006) model. One possible line of investigation would be to impose smoothness of the age, period and/or cohort parameters of the Renshaw and Haberman model using P-splines, to avoid differences between successive parameter estimates which the data do not provide statistically significant evidence to justify. It could be argued that the age parameters are the ones for which the use of P-splines is most appropriate, because of the strong prior belief that mortality should be an increasing function of age rather than fluctuating from age to age and because time series methods are then still available for forecasting future period and cohort effects.

- It would be valuable to calculate stochastic projections of future mortality rates taking account of both stochastic fluctuations and parameter risk, and hence to quantify the likely financial impact of parameter estimation errors. Only stochastic fluctuations have been considered in this thesis.

- It would be valuable to develop mortality models incorporating shocks that occur in a particular year or for lives born in a particular year, in order to allow for features such as the 1919 influenza epidemic without large residuals being required.

- Developing a Bayesian procedure whereby the estimates of the gamma parameters, and of the parameters governing both the kappa and gamma time series processes, are updated as new data become available would be of value. Simulation from the joint posterior distribution would be carried out by Markov chain Monte Carlo, as in Czado *et al*. (2005).

References

Baxter S., Should projections of mortality improvements be subject to a minimum value? Presentation given to Institute of Actuaries on 26 February 2007.

Brouhns N., Denuit M., Vermunt J. K. (2002), A Poisson log-bilinear regression approach to the construction of projected lifetables, *Insurance: Mathematics and Economics*, **31**, 373-393.

Cairns A. J. G. (2000), A discussion of parameter and model uncertainty in insurance, *Insurance: Mathematics and Economics*, **27**, 313-330.

Cairns A. J. G., Blake D., Dowd K. (2006), A two-factor model for stochastic mortality with parameter uncertainty: theory and calibration, *Journal of Risk and Insurance*, **73**, 687-718.

Cairns A. J. G., Blake D., Dowd K., Coughlan G. D., Epstein D., Khalaf-Allah M. (2008), Mortality density forecasts: an analysis of six stochastic mortality models, Pensions Institute Discussion Paper PI-0801.

Cairns A. J. G., Blake D., Dowd K., Coughlan G. D., Ong A., Epstein D., Balevich I. (2007), A quantitative comparison of stochastic mortality models using data from England & Wales and the United States, Pensions Institute Discussion Paper PI-0701.

CMI (1999), Standard tables of mortality based on the 1991-94 experiences, *Continuous Mortality Investigation Reports*, **17**, 1-228.

CMI (2002), An interim basis for adjusting the '92' series mortality projections for cohort effects, Working Paper 1.

CMI (2005), Projecting future mortality: Towards a proposal for a stochastic model, Working Paper 15.

CMI (2006), Stochastic projection methodologies: further progress and P-spline model features, example results and implications, Working Paper 20.

CMI (2007), Stochastic projection methodologies: Lee-Carter model features, example results and implications, Working Paper 25.

Currie I. D., Durban M., Eilers P. H. C. (2004), Smoothing and forecasting mortality rates, *Statistical Modelling*, **4**, 279-298.

Czado C., Delwarde A., Denuit M. (2005), Bayesian Poisson log-bilinear mortality projections, *Insurance: Mathematics and Economics*, **36**, 260-284.

de Boor C. (2001), A practical guide to splines: revised edition, Springer.

Forfar D. O., McCutcheon J. J., Wilkie A. D. (1988), On graduation by mathematical formula, *Journal of the Institute of Actuaries*, **115**, 1-149.

GAD (2006), National population projections 2004-based, Series PP2 No. 25.

Lee R. D., Carter L. R. (1992), Modeling and forecasting U.S. mortality, *Journal of the American Statistical Association*, **87**, 659-675.

LifeMetrics (2007), A toolkit for measuring and managing longevity and mortality risks, LifeMetrics Technical Document, Pension Advisory Group, JP Morgan.

Renshaw A. E., Haberman S. (2006), A cohort-based extension to the Lee-Carter model for mortality reduction factors, *Insurance: Mathematics and Economics*, **38**, 556-570.

Willets R. C. (2004), The cohort effect: insights and explanations, *British Actuarial Journal*, **10**, 833-877.

Willets R. C., Gallop A. P., Leandro P. A., Lu J. L. C., Macdonald A. S., Miller K. A., Richards S. J., Robjohns N., Ryan J. P., Waters H. R. (2004), Longevity in the 21st century, *British Actuarial Journal*, **10**, 685-832.

Appendix A: Iterative scheme for estimating the parameters of Gompertz-Makeham models

We shall describe the iterative scheme by reference to the GM(2,2) case. No new principles are involved in fitting the GM(*r*,*s*) model for any other values of *r* and *s*.

The log-likelihood function is:

$$\ell = c - \sum_x \sum_t E_{xt} \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}$$
$$+ \sum_x \sum_t D_{xt} \log\{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\},$$

where *c* is a constant.

The relevant partial derivatives of the log-likelihood are:

$$\frac{\partial \ell}{\partial \kappa_t^{(0)}} = -\sum_x E_{xt} + \sum_x D_{xt} \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-1},$$

$$\frac{\partial^2 \ell}{\partial (\kappa_t^{(0)})^2} = -\sum_x D_{xt} \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-2},$$

$$\frac{\partial \ell}{\partial \kappa_t^{(1)}} = -\sum_x E_{xt}(x-\bar{x}) + \sum_x D_{xt}(x-\bar{x}) \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-1},$$

$$\frac{\partial^2 \ell}{\partial (\kappa_t^{(1)})^2} = -\sum_x D_{xt}(x-\bar{x})^2 \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-2},$$

$$\frac{\partial \ell}{\partial \kappa_t^{(3)}} = -\sum_x E_{xt} \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]$$
$$+ \sum_x D_{xt} \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})] \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-1},$$

$$\frac{\partial^2 \ell}{\partial (\kappa_t^{(3)})^2} = -\sum_x E_{xt} \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]$$
$$+ \sum_x D_{xt} \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})] \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-1}$$
$$- \sum_x D_{xt} \exp[2\kappa_t^{(3)} + 2\kappa_t^{(4)}(x-\bar{x})] \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-2},$$

$$\frac{\partial \ell}{\partial \kappa_t^{(4)}} = -\sum_x E_{xt}(x-\bar{x}) \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]$$
$$+ \sum_x D_{xt}(x-\bar{x}) \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})] \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-1},$$

$$\frac{\partial^2 \ell}{\partial (\kappa_t^{(4)})^2} = -\sum_x E_{xt}(x-\bar{x})^2 \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]$$
$$+ \sum_x D_{xt}(x-\bar{x})^2 \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})] \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-1}$$
$$- \sum_x D_{xt}(x-\bar{x})^2 \exp[2\kappa_t^{(3)} + 2\kappa_t^{(4)}(x-\bar{x})] \{\kappa_t^{(0)} + \kappa_t^{(1)}(x-\bar{x}) + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x})]\}^{-2}.$$

Suppose that 4(*n*-1) iterations have been completed, where *n* is a positive integer, and that the estimates of the kappa parameters so obtained are $\kappa_t^{(i)} = \alpha_t^{(i)}$. Note that the $\alpha_t^{(i)}$ depend on *n*. The revised estimates of the kappa parameters after the (4*n*-3)th iteration, $\kappa_t^{(i)} = \beta_t^{(i)}$, are then defined as follows, the partial derivatives being evaluated at $\kappa_t^{(i)} = \alpha_t^{(i)}$:

$$\beta_t^{(0)} = \alpha_t^{(0)} - \frac{\dfrac{\partial \ell}{\partial \kappa_t^{(0)}}}{\dfrac{\partial^2 \ell}{\partial (\kappa_t^{(0)})^2}},$$

$$\beta_t^{(1)} = \alpha_t^{(1)},$$

$$\beta_t^{(3)} = \alpha_t^{(3)},$$

$$\beta_t^{(4)} = \alpha_t^{(4)}.$$

The revised estimates of the kappa parameters after the (4*n*-2)th iteration, $\kappa_t^{(i)} = \gamma_t^{(i)}$, are defined as follows, the partial derivatives being evaluated at $\kappa_t^{(i)} = \beta_t^{(i)}$:

$$\gamma_t^{(0)} = \beta_t^{(0)},$$

$$\gamma_t^{(1)} = \beta_t^{(1)} - \frac{\dfrac{\partial \ell}{\partial \kappa_t^{(1)}}}{\dfrac{\partial^2 \ell}{\partial (\kappa_t^{(1)})^2}},$$

$$\gamma_t^{(3)} = \beta_t^{(3)},$$

$$\gamma_t^{(4)} = \beta_t^{(4)}.$$

The revised estimates of the kappa parameters after the (4*n*-1)th iteration, $\kappa_t^{(i)} = \delta_t^{(i)}$, are defined as follows, the partial derivatives being evaluated at $\kappa_t^{(i)} = \gamma_t^{(i)}$:

$$\delta_t^{(0)} = \gamma_t^{(0)},$$

$$\delta_t^{(1)} = \gamma_t^{(1)},$$

$$\delta_t^{(3)} = \gamma_t^{(3)} - \frac{\dfrac{\partial \ell}{\partial \kappa_t^{(3)}}}{\dfrac{\partial^2 \ell}{\partial (\kappa_t^{(3)})^2}},$$

$$\delta_t^{(4)} = \gamma_t^{(4)}.$$

The revised estimates of the kappa parameters after the (4*n*)th iteration, $\kappa_t^{(i)} = \varepsilon_t^{(i)}$, are defined as follows, the partial derivatives being evaluated at $\kappa_t^{(i)} = \delta_t^{(i)}$:

$$
\begin{aligned}
\varepsilon_t^{(0)} &= \delta_t^{(0)}, \\
\varepsilon_t^{(1)} &= \delta_t^{(1)}, \\
\varepsilon_t^{(3)} &= \delta_t^{(3)}, \\
\varepsilon_t^{(4)} &= \delta_t^{(4)} - \frac{\dfrac{\partial \ell}{\partial \kappa_t^{(4)}}}{\dfrac{\partial^2 \ell}{\partial (\kappa_t^{(4)})^2}}.
\end{aligned}
$$

The criteria used to determine when to stop the iterations are described in the main text.

Appendix B: Investigations into estimating both the kappa and gamma parameters by a single iterative procedure

Under the model we are considering, he force of mortality at age *x* in calendar year *t* is:

$$\mu_{xt} = \gamma_{t-x}\{\kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]\}.$$

The log-likelihood function is:

$$\ell = a - \sum_x \sum_t E_{xt}\gamma_{t-x}\{\kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]\}$$
$$+ \sum_x \sum_t D_{xt}\log\gamma_{t-x} + \sum_x \sum_t D_{xt}\log\{\kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]\},$$

where *a* is a constant.

We shall consider two different iterative procedures for maximising this log-likelihood function, which we shall refer to as Methods A and B. In both procedures, the starting values of the kappa parameters were those estimated from the GM(1,3) model before introducing gamma parameters and the starting values of the gamma parameters were all set to 1. We shall show the results of applying Methods A and B to England and Wales male data only – similar conclusions were reached for the other two data sets. In the England and Wales male data set we consider, years of birth observed for less than five years in the data have been excluded, as was done in Chapter 3.

In both Method A and Method B, each time the gamma parameters were updated, it was done by the Newton-Raphson method, leaving the kappa parameters constant. Similarly, each time one of the series of kappa parameters was updated, it was done by the Newton-Raphson method, leaving the other series of kappa parameters and the gamma parameters constant. Thus the procedure was analogous to that described in Appendix A. The difference between Methods A and B was in the order in which the different series were updated. In Method A, between each update of the gamma parameters, all the kappa series were updated once. In Method B, the kappa series were updated repeatedly until a complete loop of updating the kappa series, from one step of updating the $\kappa^{(0)}$ parameters to another, changed none of the kappa parameters by more than 10^{-6}. Method A was stopped when a complete loop from one step of updating the gamma parameters to another changed none of the kappa or gamma parameters by more than 10^{-6}. Method B was found to be slower to converge and was stopped when such a loop changed none of the kappa or gamma parameters by more than 10^{-4}. Thus it is questionable whether the iterations have been allowed to run for long enough in the implementation of Method B. However, we know that we started the iterations with reasonable values of the parameters, as the starting values were the parameter estimates under the GM(1,3) model before introducing gamma parameters. Therefore, if we find that Method B fails to produce sensible parameter values by the time none of the kappa or gamma parameters change by more than 10^{-4}, then it is unlikely that it will produce sensible parameter values if the iterations are continued any further.

Figures B.1 and B.2 show graphs of the parameter estimates under Methods A and B respectively. These figures should be compared with Figure 3.2 for the kappa parameters and with Figure 3.7(ii) for the gamma parameters. We see that there are issues with the fitted gamma parameters in that, in both Figure B.1(v) and Figure B.2(v), the gamma parameters at the extremities of the range of years of birth are over three times greater than at the centre of the range. It does not seem plausible that such variations in the gamma parameters result from genuine cohort effects. Instead it appears that the gamma parameters are serving to widen the range of shapes of the mortality curve as a function of age available in each calendar year, which was previously restricted by the GM(1,3) structure.

At the earliest years of birth, the lives concerned only appear in the data towards the top of the age range and in the earliest calendar years of the data set. Both Figure B.1 and B.2

show low values of the $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters in these early years. Thus the high values of the gamma parameters are being compensated for by these low $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters to give mortality rates of the correct magnitude.

At the latest years of birth, the lives concerned only appear in the data towards the bottom of the age range and in the latest calendar years of the data set. Both Figure B.1 and Figure B.2 show low values of the $\kappa^{(0)}$ and $\kappa^{(3)}$ parameters in these later years. Thus the high values of the gamma parameters are being compensated for by these low $\kappa^{(0)}$ and $\kappa^{(3)}$ parameters to give mortality rates of the correct magnitude.

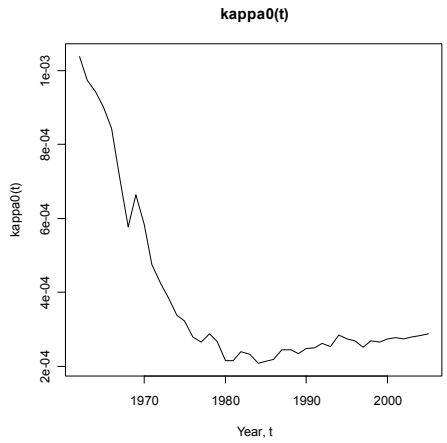
For intermediate years of birth, the gamma parameters are significantly less than 1. Mortality of the correct magnitude is achieved via high values of the $\kappa^{(0)}$ and $\kappa^{(3)}$ parameters in the early years, when these lives are towards the bottom of the age range, and via high values of the $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters in the later years, when these lives are towards the top of the age range.

Thus the reasons for the shapes of the graphs in Figures B.1 and B.2 have little to do with the underlying factors affecting mortality and naïve extrapolation of these graphs to produce future projections is unlikely to give sensible results. We would be more likely to obtain sensible results if we were adding a cohort effect to a model that already allowed for a wider range of shapes of the mortality curve as a function of age, as introducing the cohort effect would then no longer be indirectly performing the function of extending this range.

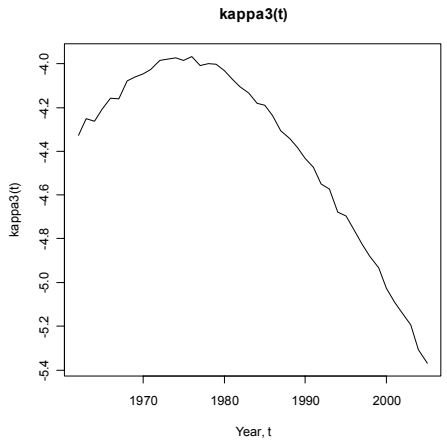
We should comment on the differences between Figures B.1 and B.2. We might have expected that both figures would be identical because both iterations should have converged to the same solution. However, we mentioned earlier that the iterations for Method B were stopped as soon as none of the parameters moved by more than 10^{-4}, which may have been too early to obtain an accurate solution from Method B. An alternative possibility is that the likelihood function has multiple maxima, but we do not have sufficient evidence to reach this conclusion. There is no reason to doubt the conclusion that it is not appropriate to calculate future projections by extrapolating parameter graphs obtained by either Method A or Method B.

Figure B.1 – Maximum likelihood parameter estimates for the GM(1,3) model extended to incorporate a cohort effect with the kappa parameters re-estimated by Method A **–** England and Wales male data **–** $\mu_{xt} = \gamma_{t-x}\{\kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]\}$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$, (v) γ_c

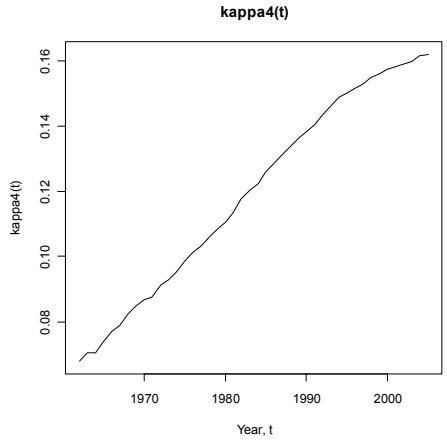
(i)



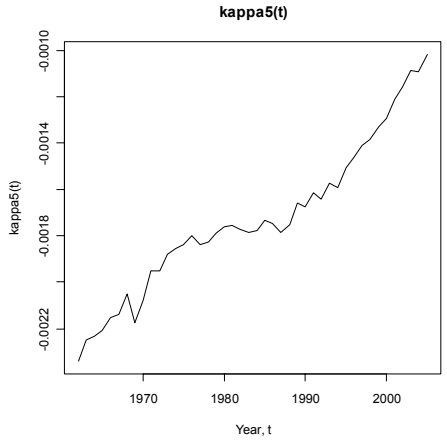
(ii)



(iii)



(iv)



(v)

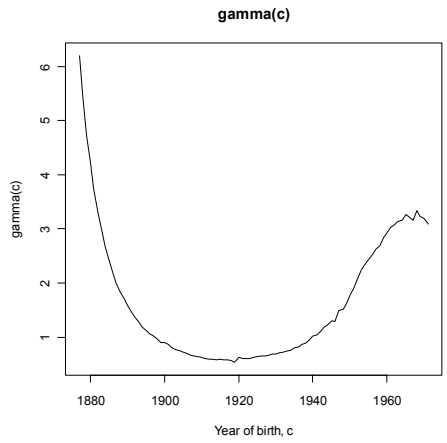
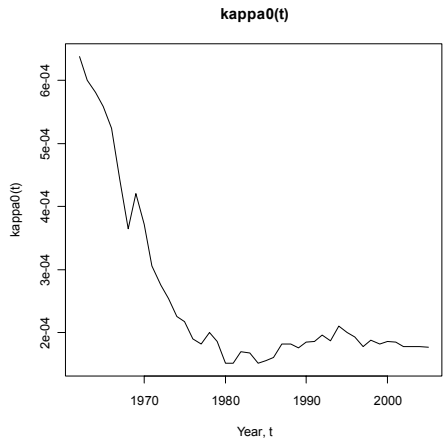
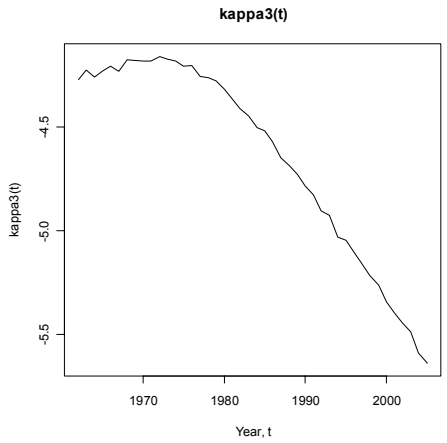


Figure B.2 – Maximum likelihood parameter estimates for the GM(1,3) model extended to incorporate a cohort effect with the kappa parameters re-estimated by Method B **–** England and Wales male data **–** $\mu_{xt} = \gamma_{t-x}\{\kappa_t^{(0)} + \exp[\kappa_t^{(3)} + \kappa_t^{(4)}(x-\bar{x}) + \kappa_t^{(5)}((x-\bar{x})^2 - \hat{\sigma}_x^2)]\}$ **–** (i) $\kappa_t^{(0)}$, (ii) $\kappa_t^{(3)}$, (iii) $\kappa_t^{(4)}$, (iv) $\kappa_t^{(5)}$, (v) γ_c

(i)



(ii)



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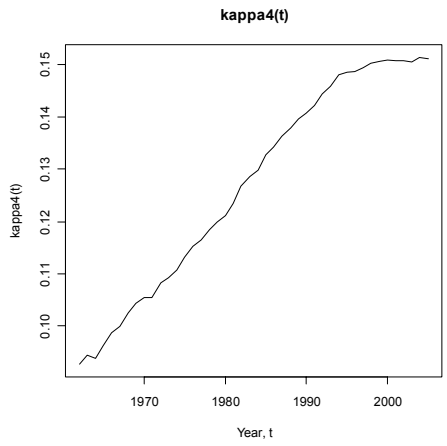
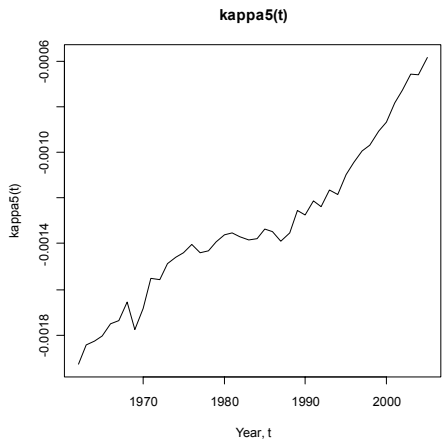
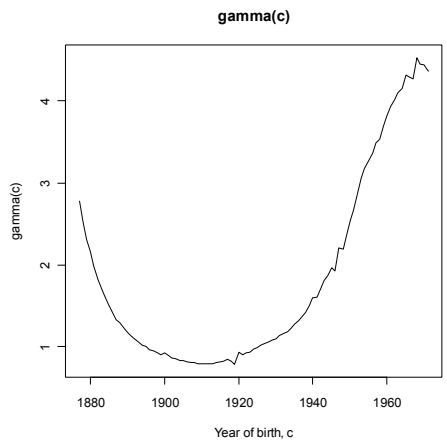


(iv)



(v)



Appendix C: Time series analysis

C.1 First-order autoregressive and moving average processes

A time series process $\{X_t\}$ is defined to be *stationary* if the distribution of X_t is independent of t.

The *autocorrelation at lag* k of a stationary time series process $\{X_t\}$ is the coefficient of correlation between X_t and X_{t-k} a s t varies, $\rho(X_t, X_{t-k})$.

As stated in Section 4.1, a *first-order autoregressive process*, or *AR(1) process*, $\{X_t\}$ is defined by:

$$X_t = \mu + \alpha(X_{t-1} - \mu) + e_t,$$

where the e_t are independent $N(0,\sigma^2)$ random variables and α, μ and σ^2 are parameters to be estimated, with $|\alpha| < 1$. The condition $|\alpha| < 1$ is imposed to ensure that the process is stationary. If this condition is not imposed, then the absolute values of the X_t will in general increase without limit as t increases.

Let $\{X_t\}$ be an AR(1) process, and let k be a positive integer. Then for each t, by repeated substitution of the defining equation into itself:

$$X_t - \mu = \alpha^k (X_{t-k} - \mu) + \sum_{i=0}^{k-1} \alpha^i e_{t-i}.$$

Taking covariances of both sides with X_{t-k} and observing that the e_u for $u > t - k$ are uncorrelated with X_{t-k}, we see that the covariance between X_t and X_{t-k} is:

$$\text{cov}(X_t, X_{t-k}) = \alpha^k \ \text{var}(X_{t-k}).$$

The autocorrelation of $\{X_t\}$ at lag k is:

$$\rho(X_t, X_{t-k}) = \frac{\text{cov}(X_t, X_{t-k})}{\sqrt{\text{var}(X_t)\,\text{var}(X_{t-k})}} = \frac{\alpha^k \ \text{var}(X_{t-k})}{\sqrt{\text{var}(X_t)\,\text{var}(X_{t-k})}}.$$

However, as the process $\{X_t\}$ is stationary, $\text{var}(X_{t-k})$ is independent of k. We therefore conclude that:

$$\rho(X_t, X_{t-k}) = \alpha^k.$$

Thus if $0 < \alpha < 1$, then the autocorrelation function of $\{X_t\}$ is positive at all lags and decays exponentially. This is consistent with the remark we made in Section 4.2 that a positive decaying autocorrelation function is characteristic of an AR(1) process. If $-1 < \alpha < 0$, then the autocorrelation function of $\{X_t\}$ alternates between positive and negative values, with the magnitude of the values decaying exponentially.

We also stated in Section 4.1 that a *first-order moving average process*, or *MA(1) process*, $\{X_t\}$ is defined by:

$$X_t = \mu + e_t + \beta e_{t-1},$$

where the e_t are independent $N(0,\sigma^2)$ random variables and β, μ and σ^2 are parameters to be estimated. It is clear from the definition that $\{X_t\}$ is stationary.

Let $\{X_t\}$ be an MA(1) process. The variance of X_t is:

$$\begin{aligned}\text{var}(X_t) &= \text{var}(\mu + e_t + \beta e_{t-1}) \\ &= \text{var}(e_t) + \beta^2\,\text{var}(e_{t-1}), \text{ since the } e_t \text{ are independent} \\ &= (1+\beta^2)\sigma^2.\end{aligned}$$

The covariance between X_t and X_{t-1} is:

$$\begin{aligned}\text{cov}(X_t, X_{t-1}) &= \text{cov}(\mu + e_t + \beta e_{t-1}, \mu + e_{t-1} + \beta e_{t-2}) \\ &= \beta\,\text{var}(e_{t-1}), \text{ since the } e_t \text{ are independent} \\ &= \beta\sigma^2.\end{aligned}$$

The autocorrelation of $\{X_t\}$ at lag 1 is therefore:

$$\rho(X_t, X_{t-1}) = \frac{\beta\sigma^2}{\sqrt{(1+\beta^2)\sigma^2(1+\beta^2)\sigma^2}} = \frac{\beta}{1+\beta^2.}.$$

For $k > 1$, the covariance between X_t and X_{t-k} is:

$$\begin{aligned}\text{cov}(X_t, X_{t-k}) &= \text{cov}(\mu + e_t + \beta e_{t-1}, \mu + e_{t-k} + \beta e_{t-k-1}) \\ &= 0, \text{ since the } e_t \text{ are independent.}\end{aligned}$$

The autocorrelation of $\{X_t\}$ at lag k is therefore also zero.

These autocorrelations are consistent with the remark we made in Section 4.3 that a characteristic feature of an MA(1) process is an autocorrelation at lag 1 which does not persist at subsequent lags.

C.2 Sample autocorrelation coefficient

Suppose we have N ordered observations, $x_1,\ldots,x_N$, from a time series process $\{X_t\}$. Let:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

For $j = 1,\ldots,N-1$, the *sample autocorrelation coefficient at lag j*, r_j, is defined as:

$$r_j = \frac{\frac{1}{N-j}\sum_{i=1}^{N-j}(x_i - \bar{x})(x_{i+j} - \bar{x})}{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}.$$

It can be shown that if $\{X_t\}$ is a pure white noise process, *i.e.* the X_t are independent identically distributed normal random variables, then r_j is approximately normally distributed with mean 0 and variance $1 / N$, for all j. We use this result to assess the significance of autocorrelations when fitting time series models in Chapter 4.

C.3 Prediction intervals

Suppose that the values $X_u = x_u$ for $u \leq t$ of an AR(1) process $\{X_t\}$ are known. We shall construct a prediction interval for the values X_{t+k} for $k > 0$. This prediction interval is used in Section 5.2.

We have:

$$X_{t+k} - \mu = \alpha^k(x_t - \mu) + \sum_{i=0}^{k-1}\alpha^i e_{t+k-i}.$$

Thus X_{t+k} is normally distributed with mean $\mu + \alpha^k(x_t - \mu)$ and variance:

$$\operatorname{var}\left(\sum_{i=0}^{k-1}\alpha^i e_{t+k-i}\right) = \sum_{i=0}^{k-1}\alpha^{2i}\operatorname{var}(e_{t+k-i}) = \sigma^2\sum_{i=0}^{k-1}\alpha^{2i} = \frac{\sigma^2(1-\alpha^{2k})}{1-\alpha^2}.$$

A 95% prediction interval for X_{t+k}, based on the 2.5th and 97.5th percentiles of the distribution of X_{t+k}, is therefore:

$$\left[\begin{array}{l}\mu + \alpha^k(x_t - \mu) - \Phi^{-1}(0.975)\sqrt{\dfrac{\sigma^2(1-\alpha^{2k})}{1-\alpha^2}}, \\ \mu + \alpha^k(x_t - \mu) + \Phi^{-1}(0.975)\sqrt{\dfrac{\sigma^2(1-\alpha^{2k})}{1-\alpha^2}}\end{array}\right],$$

where Φ is the N(0,1) distribution function.

In Chapter 5 we also use a prediction interval for future values of an ARIMA(0,1,1) process, *i.e.* a process $\{X_t\}$ such that $\{X_t - X_{t-1}\}$ is an MA(1) process. To this end, let $\{X_t\}$ be an ARIMA(0,1,1) process and suppose the values $X_u = x_u$ for $u \leq t$ are known. For each positive integer n, we have:

$$X_{t+n} - X_{t+n-1} = \mu + e_{t+n} + \beta e_{t+n-1},$$

where the e_u are independent N(0,σ^2) random variables and β, μ and σ^2 are parameters. The value of e_t is assumed to be known.

If k is a fixed positive integer, then summing the above equations for $n \leq k$ gives:

$$X_{t+k} = x_t + k\mu + e_{t+k} + (1+\beta)\sum_{i=1}^{k-1} e_{t+k-i} + \beta e_t.$$

Thus X_{t+k} is normally distributed with mean $x_t + k\mu + \beta e_t$ and variance:

$$\operatorname{var}\left(e_{t+k} + (1+\beta)\sum_{i=1}^{k-1} e_{t+k-i}\right) = \operatorname{var}(e_{t+k}) + (1+\beta)^2\sum_{i=1}^{k-1}\operatorname{var}(e_{t+k-i}) = \sigma^2(1+(k-1)(1+\beta)^2).$$

A 95% prediction interval for X_{t+k}, based on the 2.5th and 97.5th percentiles of the distribution of X_{t+k}, is therefore:

$$\begin{array}{l}[x_t + k\mu + \beta e_t - \Phi^{-1}(0.975)\sqrt{\sigma^2(1+(k-1)(1+\beta)^2)}, \\ x_t + k\mu + \beta e_t + \Phi^{-1}(0.975)\sqrt{\sigma^2(1+(k-1)(1+\beta)^2)}].\end{array}$$

Appendix D: The value of an annuity after one year

As per Section 5.6, consider an annuitant who had just attained the age label x at the start of calendar year t, where calendar year $t - 1$ is the last year of the data, and has now survived to the end of calendar year t. The (random) value of the liabilities in respect of this annuitant, $L(t + 1)$, is:

$$\sum_{i=x_1-x-1}^{\infty} v^i \exp\left(-\sum_{j=1}^{i}\mu_{x+j,t+j}\right)$$

$$= \sum_{i=x_1-x-1}^{a-x-1} v^i \exp\left(-\gamma_{t-x}\sum_{j=1}^{i}[\kappa_{t+j}^{(0)} + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(x+j-\bar{x}) + \kappa_{t+j}^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\}]\right)$$

$$+ \sum_{i=a-x}^{\infty} v^i \exp\left(\begin{array}{l} -\gamma_{t-x}\sum_{j=1}^{a-x-1}[\kappa_{t+j}^{(0)} + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(x+j-\bar{x}) + \kappa_{t+j}^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\}] \\ -\gamma_{t-x}\sum_{j=a-x}^{i}\dfrac{\mu_{a,t+j}^0\mu_{x+j,t-1}^0}{\mu_{a,t-1}^0}\end{array}\right)$$

from the high improvement assumption, where $v = 1 / 1.045$ and a is the maximum age of the data set. x_1 is the age label at which the annuity commences for a deferred annuity, and is equal to $x + 1$ for an immediate annuity.

Under the log-linear extrapolation method, for $j > a - x$, $\dfrac{\mu_{x+j,t-1}^0}{\mu_{a,t-1}^0} = \exp(\lambda(x+j-a))$, where

λ is determined so as to make $\mu_{x,t-1}^0$ a differentiable function of x at $x = a$. Therefore:

$$L(t+1)$$

$$= \sum_{i=x_1-x-1}^{a-x-1} v^i \exp\left(-\gamma_{t-x}\sum_{j=1}^{i}[\kappa_{t+j}^{(0)} + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(x+j-\bar{x}) + \kappa_{t+j}^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\}]\right)$$

$$+ \sum_{i=a-x}^{\infty} v^i \exp\left(\begin{array}{l} -\gamma_{t-x}\sum_{j=1}^{a-x-1}[\kappa_{t+j}^{(0)} + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(x+j-\bar{x}) + \kappa_{t+j}^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\}] \\ -\gamma_{t-x}\sum_{j=a-x}^{i}\exp(\lambda(x+j-a))[\kappa_{t+j}^{(0)} \\ \quad + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(a-\bar{x}) + \kappa_{t+j}^{(5)}((a-\bar{x})^2 - \hat{\sigma}_x^2)\}]\end{array}\right)$$

$$= \sum_{i=x_1-x-1}^{\infty} v^i \exp\left(-\gamma_{t-x}\sum_{j=1}^{i}A(x+j,t+j)\right),$$

where:

$$A(x+j,t+j) = \kappa_{t+j}^{(0)} + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(x+j-\bar{x}) + \kappa_{t+j}^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\}, j < a-x$$

$$= \exp(\lambda(x+j-a))[\kappa_{t+j}^{(0)} + \exp\{\kappa_{t+j}^{(3)} + \kappa_{t+j}^{(4)}(a-\bar{x}) + \kappa_{t+j}^{(5)}((a-\bar{x})^2 - \hat{\sigma}_x^2)\}], j \geq a-x.$$

Now by repeated application of the defining equations of the time series models fitted in Chapter 4, we have that for $j > 0$:

$$\kappa_{t+j}^{(i)} = \mu^{(i)} + (\alpha^{(i)})^{j+1}(\kappa_{t-1}^{(i)} - \mu^{(i)}) + \sigma^{(i)}\sum_{k=0}^{j-1}(\alpha^{(i)})^k Z_{t+j-k}^{(i)} + \sigma^{(i)}(\alpha^{(i)})^j z_t^{(i)}, i = 0,4,5,$$

$$\kappa_{t+j}^{(3)} = \kappa_{t-1}^{(3)} + (j+1)\mu^{(3)} + \sigma^{(3)}Z_{t+j}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}$$
$$+ \sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)}.$$

Note that a small letter *z* has been used for the innovations for year *t* as well as for year *t* – 1 here as these are assumed to be known by the end of year *t*.

Thus for $j < a - x$,

$$A(x+j,t+j) = \mu^{(0)} + (\alpha^{(0)})^{j+1}(\kappa_{t-1}^{(0)} - \mu^{(0)}) + \sigma^{(0)}\sum_{k=0}^{j-1}(\alpha^{(0)})^k Z_{t+j-k}^{(0)} + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}$$
$$+ \exp\left\{\begin{array}{l} \kappa_{t-1}^{(3)} + (j+1)\mu^{(3)} + \sigma^{(3)}Z_{t+j}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\ + \sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)} \\ + \left[\begin{array}{l}\mu^{(4)} + (\alpha^{(4)})^{j+1}(\kappa_{t-1}^{(4)} - \mu^{(4)}) + \sigma^{(4)}\sum_{k=0}^{j-1}(\alpha^{(4)})^k Z_{t+j-k}^{(4)} \\ + \sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}\end{array}\right](x+j-\bar{x}) \\ + \left[\begin{array}{l}\mu^{(5)} + (\alpha^{(5)})^{j+1}(\kappa_{t-1}^{(5)} - \mu^{(5)}) + \sigma^{(5)}\sum_{k=0}^{j-1}(\alpha^{(5)})^k Z_{t+j-k}^{(5)} \\ + \sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}\end{array}\right]((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\end{array}\right\}$$
$$= \mu^{(0)} + (\alpha^{(0)})^{j+1}(\kappa_{t-1}^{(0)} - \mu^{(0)}) + \sigma^{(0)}\sum_{k=0}^{j-1}(\alpha^{(0)})^k Z_{t+j-k}^{(0)} + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}$$
$$+ [1 + \sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} + \sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(x+j-\bar{x})$$
$$+ \sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)]$$
$$\exp\left\{\begin{array}{l} \kappa_{t-1}^{(3)} + (j+1)\mu^{(3)} + \sigma^{(3)}Z_{t+j}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)} + \sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)} \\ + \left[\mu^{(4)} + (\alpha^{(4)})^{j+1}(\kappa_{t-1}^{(4)} - \mu^{(4)}) + \sigma^{(4)}\sum_{k=0}^{j-1}(\alpha^{(4)})^k Z_{t+j-k}^{(4)}\right](x+j-\bar{x}) \\ + \left[\mu^{(5)} + (\alpha^{(5)})^{j+1}(\kappa_{t-1}^{(5)} - \mu^{(5)}) + \sigma^{(5)}\sum_{k=0}^{j-1}(\alpha^{(5)})^k Z_{t+j-k}^{(5)}\right]((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\end{array}\right\}$$

after linearisation in the $z_t^{(l)}$.

Define:

$$\bar{\kappa}_{t+j}^{(i)} = \mu^{(i)} + (\alpha^{(i)})^{j+1}(\kappa_{t-1}^{(i)} - \mu^{(i)}) + \sigma^{(i)}\sum_{k=0}^{j-1}(\alpha^{(i)})^k Z_{t+j-k}^{(i)}, i = 0,4,5,$$

$$\bar{\kappa}_{t+j}^{(3)} = \kappa_{t-1}^{(3)} + (j+1)\mu^{(3)} + \sigma^{(3)}Z_{t+j}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)} + \sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)}.$$

Equivalent definitions are:

$$\bar{\kappa}_{t+j}^{(i)} = \mu^{(i)} + (\alpha^{(i)})^j(\bar{\kappa}_t^{(i)} - \mu^{(i)}) + \sigma^{(i)}\sum_{k=0}^{j-1}(\alpha^{(i)})^k Z_{t+j-k}^{(i)}, i = 0,4,5,$$

$$\bar{\kappa}_{t+j}^{(3)} = \bar{\kappa}_t^{(3)} + j\mu^{(3)} + \sigma^{(3)}Z_{t+j}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)}.$$

Thus the $\bar{\kappa}_{t+j}^{(l)}$ follow the same time series processes as the $\kappa_{t+j}^{(l)}$, but starting in year *t* from the $\bar{\kappa}_t^{(l)}$, as defined in Section 5.6, rather than in year *t* – 1 from the $\kappa_{t-1}^{(l)}$, and with the innovation term $Z_t^{(3)}$ used in the MA(1) model set equal to zero. We then have:

$$\begin{aligned}
&A(x+j,t+j) = \bar{\kappa}_{t+j}^{(0)} + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)} + [1+\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\
&+ \sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(x+j-\bar{x}) + \sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)] \\
&\exp\{\bar{\kappa}_{t+j}^{(3)} + \bar{\kappa}_{t+j}^{(4)}(x+j-\bar{x}) + \bar{\kappa}_{t+j}^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)\} \\
&= \bar{\kappa}_{t+j}^{(0)} + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)} + [1+\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\
&+ \sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(x+j-\bar{x}) + \sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)]\bar{\mu}_{\exp}(x+j,t+j) \\
&= \bar{\kappa}_{t+j}^{(0)} + \bar{\mu}_{\exp}(x+j,t+j) + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)} + [\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\
&+ \sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(x+j-\bar{x}) + \sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((x+j-\bar{x})^2 - \hat{\sigma}_x^2)]\bar{\mu}_{\exp}(x+j,t+j).
\end{aligned}$$

For $j \geq a - x$,

$$\begin{aligned}
&A(x+j,t+j) = \exp(\lambda(x+j-a))\times \\
&\left[\mu^{(0)} + (\alpha^{(0)})^{j+1}(\kappa_{t-1}^{(0)} - \mu^{(0)}) + \sigma^{(0)}\sum_{k=0}^{j-1}(\alpha^{(0)})^k Z_{t+j-k}^{(0)} + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}\right. \\
&+\exp\left\{\begin{array}{l}
\kappa_{t-1}^{(3)} + (j+1)\mu^{(3)} + \sigma^{(3)}Z_{t+j}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)} + \sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\
+\sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)} \\
+\left[\begin{array}{l}\mu^{(4)} + (\alpha^{(4)})^{j+1}(\kappa_{t-1}^{(4)} - \mu^{(4)}) + \sigma^{(4)}\sum_{k=0}^{j-1}(\alpha^{(4)})^k Z_{t+j-k}^{(4)} \\ +\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}\end{array}\right](a-\bar{x}) \\
+\left[\begin{array}{l}\mu^{(5)} + (\alpha^{(5)})^{j+1}(\kappa_{t-1}^{(5)} - \mu^{(5)}) + \sigma^{(5)}\sum_{k=0}^{j-1}(\alpha^{(5)})^k Z_{t+j-k}^{(5)} \\ +\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}\end{array}\right]((a-\bar{x})^2 - \hat{\sigma}_x^2)
\end{array}\right\}\left.\right] \\
&= \exp(\lambda(x+j-a))\times \\
&\left[\mu^{(0)} + (\alpha^{(0)})^{j+1}(\kappa_{t-1}^{(0)} - \mu^{(0)}) + \sigma^{(0)}\sum_{k=0}^{j-1}(\alpha^{(0)})^k Z_{t+j-k}^{(0)} + \sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}\right] +
\end{aligned}$$

$$+\exp(\lambda(x+j-a))\times$$
$$[1+\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}+\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(a-\bar{x})+\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)]$$
$$\exp\left\{\begin{array}{l}\kappa_{t-1}^{(3)}+(j+1)\mu^{(3)}+\sigma^{(3)}Z_{t+j}^{(3)}+\sigma^{(3)}(1+\beta^{(3)})\sum_{k=1}^{j-1}Z_{t+j-k}^{(3)}+\sigma^{(3)}\beta^{(3)}z_{t-1}^{(3)}\\ +\left[\mu^{(4)}+(\alpha^{(4)})^{j+1}(\kappa_{t-1}^{(4)}-\mu^{(4)})+\sigma^{(4)}\sum_{k=0}^{j-1}(\alpha^{(4)})^k Z_{t+j-k}^{(4)}\right](a-\bar{x})\\ +\left[\mu^{(5)}+(\alpha^{(5)})^{j+1}(\kappa_{t-1}^{(5)}-\mu^{(5)})+\sigma^{(5)}\sum_{k=0}^{j-1}(\alpha^{(5)})^k Z_{t+j-k}^{(5)}\right]((a-\bar{x})^2-\hat{\sigma}_x^2)\end{array}\right\}$$

after linearisation in the $z_t^{(l)}$

$$=\exp(\lambda(x+j-a))\left[\bar{\kappa}_{t+j}^{(0)}+\sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}\right]+\exp(\lambda(x+j-a))\times$$
$$[1+\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}+\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(a-\bar{x})+\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)]$$
$$\exp\left\{\bar{\kappa}_{t+j}^{(3)}+\bar{\kappa}_{t+j}^{(4)}(a-\bar{x})+\bar{\kappa}_{t+j}^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)\right\}$$
$$=\exp(\lambda(x+j-a))\left[\bar{\kappa}_{t+j}^{(0)}+\sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}\right]$$
$$+[1+\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}+\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(a-\bar{x})+\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)]$$
$$\bar{\mu}_{\exp}(x+j,t+j)$$

where we define, for $j \geq a-x$,

$$\bar{\mu}_{\exp}(x+j,t+j)=\exp(\lambda(x+j-a))\exp\left\{\bar{\kappa}_{t+j}^{(3)}+\bar{\kappa}_{t+j}^{(4)}(a-\bar{x})+\bar{\kappa}_{t+j}^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)\right\}$$

Thus:

$$A(x+j,t+j)=\exp(\lambda(x+j-a))\bar{\kappa}_{t+j}^{(0)}+\bar{\mu}_{\exp}(x+j,t+j)$$
$$+\exp(\lambda(x+j-a))\sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}$$
$$+[\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}+\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(a-\bar{x})+\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)]$$
$$\bar{\mu}_{\exp}(x+j,t+j).$$

Substituting these expressions for $A(x+j,t+j)$ into our formula for $L(t+1)$ gives $L(t+1)$ as:

$$\sum_{i=x_1-x-1}^{a-x-1}v^i\exp\left(-\gamma_{t-x}\sum_{j=1}^{i}\left\{\begin{array}{l}\bar{\kappa}_{t+j}^{(0)}+\bar{\mu}_{\exp}(x+j,t+j)+\sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}+[\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\\ +\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(x+j-\bar{x})\\ +\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((x+j-\bar{x})^2-\hat{\sigma}_x^2)]\bar{\mu}_{\exp}(x+j,t+j)\end{array}\right\}\right)+$$

$$+\sum_{i=a-x}^{\infty} v^i \exp\left(\begin{array}{l} -\gamma_{t-x}\sum_{j=1}^{a-x-1}\left\{\begin{array}{l}\overline{\kappa}_{t+j}^{(0)}+\overline{\mu}_{\exp}(x+j,t+j)+\sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}+[\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\ +\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(x+j-\overline{x}) \\ +\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((x+j-\overline{x})^2-\hat{\sigma}_x^2)]\overline{\mu}_{\exp}(x+j,t+j)\end{array}\right\} \\ -\gamma_{t-x}\sum_{j=a-x}^{i}\left\{\begin{array}{l}\exp(\lambda(x+j-a))\overline{\kappa}_{t+j}^{(0)}+\overline{\mu}_{\exp}(x+j,t+j) \\ +\exp(\lambda(x+j-a))\sigma^{(0)}(\alpha^{(0)})^j z_t^{(0)}+[\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)} \\ +\sigma^{(4)}(\alpha^{(4)})^j z_t^{(4)}(a-\overline{x})+\sigma^{(5)}(\alpha^{(5)})^j z_t^{(5)}((a-\overline{x})^2-\hat{\sigma}_x^2)] \\ \overline{\mu}_{\exp}(x+j,t+j)\end{array}\right\}\end{array}\right)$$

$$=\sum_{i=x_1-x-1}^{a-x-1} v^i \exp\left(\begin{array}{l} -\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}-\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\sum_{j=1}^{i}(\alpha^{(0)})^j \\ -\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\sum_{j=1}^{i}\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(4)}z_t^{(4)}\sum_{j=1}^{i}(\alpha^{(4)})^j(x+j-\overline{x})\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(5)}z_t^{(5)}\sum_{j=1}^{i}(\alpha^{(5)})^j((x+j-\overline{x})^2-\hat{\sigma}_x^2)\overline{\mu}_{\exp}(x+j,t+j)\end{array}\right)$$

$$+\sum_{i=a-x}^{\infty} v^i \exp\left(\begin{array}{l} -\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}-\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\sum_{j=1}^{a-x-1}(\alpha^{(0)})^j \\ -\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\sum_{j=1}^{a-x-1}\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(4)}z_t^{(4)}\sum_{j=1}^{a-x-1}(\alpha^{(4)})^j(x+j-\overline{x})\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(5)}z_t^{(5)}\sum_{j=1}^{a-x-1}(\alpha^{(5)})^j((x+j-\overline{x})^2-\hat{\sigma}_x^2)\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\sum_{j=a-x}^{i}(\alpha^{(0)})^j\exp(\lambda(x+j-a)) \\ -\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\sum_{j=a-x}^{i}\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(4)}z_t^{(4)}(a-\overline{x})\sum_{j=a-x}^{i}(\alpha^{(4)})^j\overline{\mu}_{\exp}(x+j,t+j) \\ -\gamma_{t-x}\sigma^{(5)}z_t^{(5)}((a-\overline{x})^2-\hat{\sigma}_x^2)\sum_{j=a-x}^{i}(\alpha^{(5)})^j\overline{\mu}_{\exp}(x+j,t+j)\end{array}\right)$$

where we define, for $j \geq a-x$,

$$\overline{\mu}_{x+j,t+j} = \gamma_{t-x}[\exp(\lambda(x+j-a))\overline{\kappa}_{t+j}^{(0)} + \overline{\mu}_{\exp}(x+j,t+j)].$$

Thus:

$$L(t+1)=\sum_{i=x_1-x-1}^{a-x-1}v^i\exp\begin{pmatrix}-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}-\dfrac{\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\alpha^{(0)}(1-(\alpha^{(0)})^i)}{1-\alpha^{(0)}}\\-\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\sum_{j=1}^{i}\overline{\mu}_{\exp}(x+j,t+j)\\-\gamma_{t-x}\sigma^{(4)}z_t^{(4)}\sum_{j=1}^{i}(\alpha^{(4)})^j(x+j-\bar{x})\overline{\mu}_{\exp}(x+j,t+j)\\-\gamma_{t-x}\sigma^{(5)}z_t^{(5)}\sum_{j=1}^{i}(\alpha^{(5)})^j((x+j-\bar{x})^2-\hat{\sigma}_x^2)\overline{\mu}_{\exp}(x+j,t+j)\end{pmatrix}$$

$$+\sum_{i=a-x}^{\infty}v^i\exp\begin{pmatrix}-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}-\dfrac{\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}\\-\dfrac{\gamma_{t-x}\sigma^{(0)}z_t^{(0)}(\alpha^{(0)})^{a-x}(1-(\alpha^{(0)}e^{\lambda})^{i-a+x+1})}{1-\alpha^{(0)}e^{\lambda}}\\-\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\sum_{j=1}^{i}\overline{\mu}_{\exp}(x+j,t+j)\\-\gamma_{t-x}\sigma^{(4)}z_t^{(4)}\sum_{j=1}^{a-x-1}(\alpha^{(4)})^j(x+j-\bar{x})\overline{\mu}_{\exp}(x+j,t+j)\\-\gamma_{t-x}\sigma^{(4)}z_t^{(4)}(a-\bar{x})\sum_{j=a-x}^{i}(\alpha^{(4)})^j\overline{\mu}_{\exp}(x+j,t+j)\\-\gamma_{t-x}\sigma^{(5)}z_t^{(5)}\sum_{j=1}^{a-x-1}(\alpha^{(5)})^j((x+j-\bar{x})^2-\hat{\sigma}_x^2)\overline{\mu}_{\exp}(x+j,t+j)\\-\gamma_{t-x}\sigma^{(5)}z_t^{(5)}((a-\bar{x})^2-\hat{\sigma}_x^2)\sum_{j=a-x}^{i}(\alpha^{(5)})^j\overline{\mu}_{\exp}(x+j,t+j)\end{pmatrix}$$

$$=\sum_{i=x_1-x-1}^{\infty}v^i\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)-\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}z_t^{(0)}}{1-\alpha^{(0)}}\sum_{i=x_1-x-1}^{a-x-1}v^i\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}z_t^{(0)}}{1-\alpha^{(0)}}\sum_{i=x_1-x-1}^{a-x-1}(v\alpha^{(0)})^i\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$-\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}+\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right)\sum_{i=a-x}^{\infty}v^i\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}z_t^{(0)}}{1-\alpha^{(0)}e^{\lambda}}\sum_{i=a-x}^{\infty}(v\alpha^{(0)}e^{\lambda})^i\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$-\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}\sum_{i=x_1-x-1}^{\infty}v^i\left(\sum_{j=1}^{i}\overline{\mu}_{\exp}(x+j,t+j)\right)\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$-\gamma_{t-x}\sigma^{(4)}z_t^{(4)}\sum_{i=x_1-x-1}^{a-x-1}v^i\left(\sum_{j=1}^{i}(\alpha^{(4)})^j(x+j-\bar{x})\overline{\mu}_{\exp}(x+j,t+j)\right)-$$

$$\exp\left(-\sum_{j=1}^{i}\overline{\mu}_{x+j,t+j}\right)$$

$$
\begin{aligned}
&- \gamma_{t-x}\sigma^{(4)} z_t^{(4)} \sum_{i=a-x}^{\infty} v^i \left(\begin{array}{l} \sum_{j=1}^{a-x-1} (\alpha^{(4)})^j (x+j-\bar{x}) \bar{\mu}_{\exp}(x+j,t+j) \\ + (a-\bar{x}) \sum_{j=a-x}^{i} (\alpha^{(4)})^j \bar{\mu}_{\exp}(x+j,t+j) \end{array} \right) \\
&\qquad \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right) \\
&- \gamma_{t-x}\sigma^{(5)} z_t^{(5)} \sum_{i=x_1-x-1}^{a-x-1} v^i \left(\sum_{j=1}^{i} (\alpha^{(5)})^j ((x+j-\bar{x})^2 - \hat{\sigma}_x^2) \bar{\mu}_{\exp}(x+j,t+j) \right) \\
&\qquad \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right) \\
&- \gamma_{t-x}\sigma^{(5)} z_t^{(5)} \sum_{i=a-x}^{\infty} v^i \left(\begin{array}{l} \sum_{j=1}^{a-x-1} (\alpha^{(5)})^j ((x+j-\bar{x})^2 - \hat{\sigma}_x^2) \bar{\mu}_{\exp}(x+j,t+j) \\ + ((a-\bar{x})^2 - \hat{\sigma}_x^2) \sum_{j=a-x}^{i} (\alpha^{(5)})^j \bar{\mu}_{\exp}(x+j,t+j) \end{array} \right) \\
&\qquad \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right)
\end{aligned}
$$

after linearisation in the $z_t^{(l)}$.

Note that the summation terms over the index *i* are all independent of the $z_t^{(l)}$ and have the form of annuity values. Some of the annuities have payments that vary over time, and others are valued at revised interest rates. We now introduce the following notation for the summation terms over the index *i*:

$$
\begin{aligned}
FutAnnVal(x,t) &= \sum_{i=x_1-x-1}^{\infty} v^i \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right), \\
FutAnnVal\_Maxa(x,t) &= \sum_{i=x_1-x-1}^{a-x-1} v^i \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right), \\
FutAnnVal\_Maxa\_alpha0(x,t) &= \sum_{i=x_1-x-1}^{a-x-1} (v\alpha^{(0)})^i \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right), \\
FutAnnVal\_Mina\_alpha0\_lambda(x,t) &= \sum_{i=a-x}^{\infty} (v\alpha^{(0)} e^{\lambda})^i \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right), \\
FutAnnVal\_Varying3(x,t) &= \sum_{i=x_1-x-1}^{\infty} v^i \left(\sum_{j=1}^{i} \bar{\mu}_{\exp}(x+j,t+j) \right) \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right), \\
FutAnnVal\_Varying4(x,t) &= \sum_{i=x_1-x-1}^{a-x-1} v^i \left(\sum_{j=1}^{i} (\alpha^{(4)})^j (x+j-\bar{x}) \bar{\mu}_{\exp}(x+j,t+j) \right) + \\
&\qquad \exp\left(- \sum_{j=1}^{i} \bar{\mu}_{x+j,t+j} \right)
\end{aligned}
$$

$$+\sum_{i=a-x}^{\infty} \frac{v^i\left(\sum_{j=1}^{a-x-1}(\alpha^{(4)})^j(x+j-\bar{x})\bar{\mu}_{\exp}(x+j,t+j)+(a-\bar{x})\sum_{j=a-x}^{i}(\alpha^{(4)})^j\bar{\mu}_{\exp}(x+j,t+j)\right)}{\exp\left(-\sum_{j=1}^{i}\bar{\mu}_{x+j,t+j}\right)},$$

$$FutAnnVal\_Varying5(x,t)=\sum_{i=x_1-x-1}^{a-x-1}\frac{v^i\left(\sum_{j=1}^{i}(\alpha^{(5)})^j((x+j-\bar{x})^2-\hat{\sigma}_x^2)\bar{\mu}_{\exp}(x+j,t+j)\right)}{\exp\left(-\sum_{j=1}^{i}\bar{\mu}_{x+j,t+j}\right)}$$

$$+\sum_{i=a-x}^{\infty}\frac{v^i\left(\begin{array}{l}\sum_{j=1}^{a-x-1}(\alpha^{(5)})^j((x+j-\bar{x})^2-\hat{\sigma}_x^2)\bar{\mu}_{\exp}(x+j,t+j)\\+((a-\bar{x})^2-\hat{\sigma}_x^2)\sum_{j=a-x}^{i}(\alpha^{(5)})^j\bar{\mu}_{\exp}(x+j,t+j)\end{array}\right)}{\exp\left(-\sum_{j=1}^{i}\bar{\mu}_{x+j,t+j}\right)}.$$

Then:

$$L(t+1)=FutAnnVal(x,t)-\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}z_t^{(0)}}{1-\alpha^{(0)}}FutAnnVal\_Maxa(x,t)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}z_t^{(0)}}{1-\alpha^{(0)}}FutAnnVal\_Maxa\_alpha0(x,t)$$

$$-\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}+\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right)$$

$$[FutAnnVal(x,t)-FutAnnVal\_Maxa(x,t)]$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}z_t^{(0)}}{1-\alpha^{(0)}e^{\lambda}}FutAnnVal\_Mina\_alpha0\_lambda(x,t)$$

$$-\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}FutAnnVal\_Varying3(x,t)$$

$$-\gamma_{t-x}\sigma^{(4)}z_t^{(4)}FutAnnVal\_Varying4(x,t)-\gamma_{t-x}\sigma^{(5)}z_t^{(5)}FutAnnVal\_Varying5(x,t)$$

$$=FutAnnVal(x,t)+\frac{\gamma_{t-x}\sigma^{(0)}(\alpha^{(0)})^{a-x+1}(e^{\lambda}-1)z_t^{(0)}}{(1-\alpha^{(0)})(1-\alpha^{(0)}e^{\lambda})}FutAnnVal\_Maxa(x,t)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}z_t^{(0)}}{1-\alpha^{(0)}}FutAnnVal\_Maxa\_alpha0(x,t)-$$

$$-\gamma_{t-x}\sigma^{(0)}z_t^{(0)}\left(\frac{\alpha^{(0)}(1-(\alpha^{(0)})^{a-x-1})}{1-\alpha^{(0)}}+\frac{(\alpha^{(0)})^{a-x}}{1-\alpha^{(0)}e^{\lambda}}\right)FutAnnVal(x,t)$$

$$+\frac{\gamma_{t-x}\sigma^{(0)}\alpha^{(0)}e^{-(a-x-1)\lambda}z_t^{(0)}}{1-\alpha^{(0)}e^{\lambda}}FutAnnVal\_Mina\_alpha0\_lambda(x,t)$$

$$-\gamma_{t-x}\sigma^{(3)}(1+\beta^{(3)})z_t^{(3)}FutAnnVal\_Varying3(x,t)$$

$$-\gamma_{t-x}\sigma^{(4)}z_t^{(4)}FutAnnVal\_Varying4(x,t)-\gamma_{t-x}\sigma^{(5)}z_t^{(5)}FutAnnVal\_Varying5(x,t).$$