JIA 111 (1984) 445-501

THE MATCHING OF ASSETS TO LIABILITIES

By A. J. Wise, M.A., F.I.A., F.S.S., A.P.M.I.

[Submitted to the Institute, 26th March 1984]

Theories are nets cast to catch what we call 'the world': to rationalise, to explain, and to master it. We endeavour to make the mesh ever finer and finer.

Professor Sir Karl Popper in *The Logic of Scientific Discovery*

1. INTRODUCTION

1.1. The concept of the matching of assets to liabilities is fundamental in matters of finance. In its broadest sense matching is relevant both to the investment of life office and pension funds and to actuarial calculations in relation to those funds.

1.2. Matching is inherently connected with the uncertainty of future conditions, as measured by such indicators as rates of interest, inflation and currency exchange. It is when conditions do not work out as hoped for or expected that the risks associated with mis-matching can materialize. Whether considering investment policy or an actuarial calculation it is important that the nature of a portfolio which minimizes these risks be appreciated, and yet it is often difficult to specify such a portfolio with precision. This may not be a source of difficulty in practice when the judgement of the actuary is applied to the problem in hand, but there is evident scope for refinement of the concepts.

1.3. The purpose of this paper is to present the results of a new study in which the matching position is well defined by reference to appropriate actuarial models. The new theory leads to specific portfolio structures which comprise fixed interest and equity or index-linked investments and which, in a defined sense, are the best match to the given liabilities. As will be shown, the advantages of this approach emerge in a variety of applications, especially to problems of pension funds. In particular it is found possible to quantify aspects of actuarial valuation which would otherwise only be considered in the light of general reasoning.

2. BACKGROUND AND DEFINITIONS

2.1. The new approach which is the subject of this paper emerges from the juxtaposition of certain key ideas, none of which in itself is novel. We shall review the background and draw together the ideas as we go. For ease of reference a glossary of defined terms is given in the Appendix.

Absolute matching

2.2. If the assets of a fund are so arranged that the future receipts of interest and capital are certain to match precisely, both in amount and timing, all the future net expenditure of the fund, then variations in future economic conditions cannot affect its long-term financial position. This is absolute matching, as first discussed by Haynes and Kirton[1]. The concept of absolute matching is an ideal which is unlikely to be realized in practice, if only because of imperfections and uncertainties in the market for available fixed interest stocks. For most funds the nature of the liabilities will rule out the possibility of matching to anywhere near this ideal extent. Nevertheless, this ideal plays an important role in the new theory. (See § 4.5).

Immunization

2.3. The idea of matching to protect a fund against changes in interest rates has been well researched in terms of immunization for a life office. The immunization conditions as stated by Redington[2] will test whether a fund is protected against developing a deficiency in the event of a general change in interest rates. The conditions are admirable in their value and practicality but the mathematical model from which they are derived is a simple one, in effect that of a small instantaneous change in the interest rate and no further change thereafter. The theory of immunization has therefore been developed further in various directions. Shedden[3] obtained more general conditions for immunization allowing for, among other factors, further changes in the interest rate. Boyle[4] gave conditions for immunization under stochastic models of the interest rate. Tilley[5] considered the range of portfolios which immunize against a specified determinate structure of future interest rates. However the theory as extended remains one of application principally to life office business where the objective is to protect against either a deficiency or a reduction of bonus rate.

Pension schemes

2.4 It is sometimes said that matching is of little relevance to final salary pension schemes because of the nature and term of their liabilities. This might be true if matching were synonymous with immunization, but a clear distinction should be drawn between these two concepts. Fellows[6] discussed matching for pension funds in a recent paper; we shall approach the subject differently and return to some of his points later.

2.5 Keeping for the moment to fundamentals, what is meant by matching in the context of a pension fund? There are closed pension funds with fixed monetary liabilities which have features in common with life office funds. Matching by fixed interest investments with appropriate terms to maturity may well be a proper investment objective for such funds. Most pension schemes in the U.K. are now of the final salary configuration, and the predominance of liabilities which are tied in some way to future rates of inflation is regarded as justifying investment in 'real assets' such as equity shares and property. This is a matching argument of another kind.

2.6. Pension funds are dynamic entities and the nature of their liabilities changes with economic and social conditions in the country as well as their own growing maturity. The last few years have witnessed massive increases in liabilities through inflation. More recently the experience in many pension funds has been a fall in the active membership resulting in a significant shift in the weight of liabilities from accruing to vested benefits. A comprehensive concept of matching for pension funds would need to encompass these changes.

The closed fund

2.7. It is a widely accepted though not universal principle that an actuarial valuation of a pension scheme or life office fund should be made as if the fund were closed to new entrants, so as to exclude from the valuation any possible cross-subsidy between present and future participants in the fund. The closed fund principle is fundamental to the ideas in this paper. We shall be regarding any fund as if it were closed to new entrants, whether it is actually closed or not. It is the adoption of this principle which leads to greater precision in the concept of matching. The consequence of regarding a fund as closed to new entrants is that its lifetime becomes finite and we may consider the effects of alternative future economic conditions in terms of the ultimate surplus or deficiency which results at the end of that time.

Ultimate surplus

2.8. The ultimate surplus which results in a closed fund can be measured in terms of the realizable market value of the assets remaining when all liabilities have been extinguished. A deficiency would emerge in the form of outstanding borrowings to cover the final liability payments, and this too can be quantified in cash terms and regarded as a negative surplus.

2.9 Viewing the life office or pension fund at any one time, an actuarial valuation on a closed fund basis typically seeks to determine the present value of the ultimate surplus. Information about the matching position and the degree of vulnerability to future changes in economic conditions can be gauged by making further valuations at the same date using alternative assumptions. It is possible to develop further this approach to matching, but in order to do so it is necessary to abandon, at least temporarily, the concept of present value and concentrate instead on the ultimate surplus.

Stochastic models

2.10. The next step is to introduce a greater element of realism into our actuarial models. It has already been noted that matching is inherently connected with uncertainty of future conditions. To make headway, we should extend our models to represent not only the average expected rates of interest and inflation in future years, but also the probability distributions of these factors. In principle we could also seek to model potentially volatile demographic factors such as lapse or withdrawal rates in this way, and indeed it might be necessary to do so if we were dealing with small numbers of people. However in this paper demographic factors will be treated as fixed parameters.

2.11. With such a stochastic model of interest and inflation, the ultimate surplus takes on the character of a random variable with a probability distribution. Within the framework of this model we can envisage different realizations leading to different values of ultimate surplus. Moreover any realization of future conditions will reflect not only the future rates of interest and inflation which develop but also the characteristics of the investment portfolio. In order to develop this idea we disregard any future switching of investments and assume that the present investments are held until their redemption or earlier enforced sale to meet the liabilities. On this basis the probability distribution of the ultimate surplus will depend upon the present portfolio, and the nature of that distribution can be controlled to some extent by selection of appropriate portfolios from the market of available investments.

Portfolio selection

2.12. Moore[7] described the theory of portfolio selection as developed from the work of Markowitz. This theory is based on stochastic investment models and is concerned with the selection of portfolios of maximum expected return or minimum variance of return. As presented the theory is not concerned with matching, because the investment return is measured over a specific period and is not related to the emergence with time of the liabilities of the investor.

2.13. However we can now relate the selection of portfolios to the liabilities by consideration of the mean and variation of the ultimate surplus. Supposing that we had a free choice from all possible portfolios for the present fund, without even any restriction as to size, what would be the relationship between the ultimate surplus and the choice of portfolio? Obviously the portfolios with the larger market values at the outset are more likely to lead to larger surpluses. Relatively small portfolios will produce deficiencies. Portfolios which are in some way badly matched to the liabilities are likely to produce the greatest variation of ultimate surplus. If the liabilities could be matched absolutely, the ultimate surplus resulting from the matching portfolio would be zero, without variation.

Matching portfolios

2.14. In view of this, the portfolios which give rise to a mean ultimate surplus of zero and minimum variation of surplus are clearly of importance in the study of matching. Any such portfolio will be referred to as a 'positive unbiased match'. The absolute match is one example of this class, but one which will not be attainable unless the liability cash flows form the right sort of pattern in relation to the investments which are available in the market.

2.15. The term 'positive' refers here to the exclusion of negative holdings of assets, namely borrowings, from the admissible portfolios. (Future borrowings, essentially of a short-term nature, will be allowed to cover future cash flow requirements.) The term 'unbiased' refers to the condition of the ultimate surplus having a mean of zero.

2.16. In practice it transpires that we can drop the unbiased restriction and focus attention on the class of portfolios which give rise to minimum variation of ultimate surplus about zero. Any such portfolio is referred to for the time being as a 'positive match'. The mean ultimate surplus arising from a positive match may be non-zero, thus producing a bias, but the mean cannot be greatly different from zero because it will figure in the variation which is to be minimized of surplus about zero. The numerical results shown later in this paper support this point. (See § 5.18.)

2.17. To summarize, a positive match is defined as a portfolio which minimizes the mean square ultimate surplus. Being expressed in purely mathematical terms, the problem of finding a positive match for any given actuarial model is one which can be investigated mathematically.

The main result

2.18. A summary of the mathematical analysis follows in § 4. However the main result can now be stated in outline without further delay. Given:

(i) any suitable actuarial model of future conditions in which the demographic factors are fixed and the factors of interest and inflation follow a specified statistical behaviour;
(ii) a suitable model of the investments available in the market and of the future cash flows which each is expected to generate; and
(iii) any pattern of liability cash flows,

then there exists one and only one positive match to the given liabilities. There is an explicit formulation for the mathematical function which assigns, to each and every possible pattern of liabilities, its unique positive match. Matching can therefore be defined in a generalized way, in relation to any such actuarial model, in terms of this assignment of portfolios to liabilities.

2.19. It is interesting to compare this idea of matching with the distinct theories of immunization and of portfolio selection, to which reference has already been made. The three theories are concerned with different objectives and their broad characteristics are compared in the following table:

	Objective in relation to uncertainty of future conditions	Number of possible portfolios which meet the objective in any specific case
Immunization	Protect against deficiency	Zero or infinite
Portfolio selection	Minimize variance of outcome of investment, or maximize expected outcome for given variance, disregarding liabilities	Infinite
Matching	Minimize mean square ultimate surplus	One

2.20. In certain respects matching theory fits between the other two: it resembles the theory of portfolio selection in seeking to minimize uncertainty of outcome but it also brings into account the liability cash flows in a similar way to immunization.

3. A WORKED EXAMPLE

3.1. A worked example will help to fix ideas and introduce notation and formulae. Let us construct an elementary actuarial model in which all cash flows occur at the ends of years 1, 2 and 3, and where there are no demographic factors and no inflation.

Actuarial model

3.2. The only relevant factor is the rate of investment return during each of the three years, which we shall treat as a deposit rate. Assume for simplicity that each rate can only take one of two values, 8% or 10% with equal probability, and that there is no correlation in the interest rate from one year to the next. We thus have three independent random variables in the model, and in fact only the rates in years 2 and 3 will be relevant to this example.

3.3. We shall need to know the result of accumulating unit cash flow at time t with interest up to time 3. Call this quantity F_t. We can enumerate all possible values of F_t and their respective probabilities as in the following table.

	Value	Probability
F_1	$(1{\cdot}08)^2$	·25
	$1{\cdot}08 \times 1{\cdot}10$	·50
	$(1{\cdot}10)^2$	·25
F_2	1·08	·50
	1·10	·50
F_3	1·0	1·0

3.4. For the investment model suppose there are just two basic stocks available in the market, each yielding an income of 10% payable annually in arrears. One stock matures at the end of year 2 and the other at the end of year 3, and a total payment of 1 (including interest) is made on each redemption date. The cash flows resulting from a unit holding of each of these stocks may be represented by row vectors thus:

$$\mathbf{e}_1 = (\cdot 1 \quad 1 \quad 0)$$
$$\mathbf{e}_2 = (\cdot 1 \quad \cdot 1 \quad 1)$$

More concisely, the available investments can be expressed in terms of a matrix $\mathbf{E}$ whose rows are the cash flows of the basic assets. In this example:

$$\mathbf{E} = \begin{pmatrix} \cdot 1 & 1 & 0 \\ \cdot 1 & \cdot 1 & 1 \end{pmatrix}$$

3.5. Any portfolio can now be expressed as a combination of the basic assets, and the resulting cash flows can be represented by a further vector $\mathbf{a} = (a_1, a_2, a_3)$. In this example with just two basic assets the general form of $\mathbf{a}$ is:

$$\mathbf{a} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$$

which may also be written in the form

$$\mathbf{a} = \mathbf{x}\,\mathbf{E}$$

where

$$\mathbf{x} = (x_1, x_2)$$

3.6. Finally the liabilities must be specified in terms of cash flows at the three year ends. Suppose the outgo is of unit amount in each year. The liabilities can be represented by a vector in the same format as above:

$$\mathbf{l} = (l_1, l_2, l_3)$$

where in this case l_1, l_2, l_3 are all unity.

Algebraic solution

3.7. The objective is to evaluate the ultimate surplus which results from any portfolio $\mathbf{a}$ and to find the portfolio which minimizes the mean square surplus. In the first place we shall use algebra and insert the numerical values in the final formula.

3.8. The ultimate surplus is:

$$S = (a_1 - l_1)\,F_1 + (a_2 - l_2)\,F_2 + (a_3 - l_3)\,F_3$$

F_1 and F_2 are random variables as set out in the above table, so the ultimate surplus S is also a random variable. The mean square surplus, which will be called E_2, is the expectation of S^2. In this quantity E_2 the coefficient of $(a_i - l_i)(a_j - l_j)$ is the expectation of $F_i\,F_j$, which we shall call C_{ij}.

Thus

$$E_2 = \sum_{i,j} (a_i - l_i)\,C_{ij}\,(a_j - l_j)$$

which may be written

$$E_2 = (\mathbf{a} - \mathbf{l})\,\mathbf{C}\,(\mathbf{a} - \mathbf{l})'$$

where $\mathbf{C}$ is the covariance type matrix with elements C_{ij}, and dash denotes a transpose or in this context a column vector.

3.9. In our simple example the elements C_{ij} can be calculated by enumerating all possible cases of future interest rates, as shown in the following table.

Interest rates % Year 2	Year 3	F_1 $(=F_1F_3)$	F_2 $(=F_2F_3)$	F_1^2	F_2^2	F_1F_2
8	8	1·1664	1·08	1·3605	1·1664	1·2597
8	10	1·188	1·10	1·4113	1·21	1·3068
10	8	1·188	1·08	1·4113	1·1664	1·2830
10	10	1·21	1·10	1·4641	1·21	1·3310
Mean		1·1881	1·09	1·4118	1·1882	1·2951

Thus

$$\mathbf{C} = \begin{pmatrix} 1{\cdot}4118 & 1{\cdot}2951 & 1{\cdot}1881 \\ 1{\cdot}2951 & 1{\cdot}1882 & 1{\cdot}09 \\ 1{\cdot}1881 & 1{\cdot}09 & 1{\cdot}0 \end{pmatrix}$$

3.10. Expressing **a** in terms of the basic assets:

$$E_2 = (x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 - \mathbf{l})\,\mathbf{C}\,(x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 - \mathbf{l})'$$

where everything has been evaluated except the asset proportions x_1 and x_2. To find the proportions which minimize E_2 we require:

$$\frac{\partial E_2}{\partial x_1} = \frac{\partial E_2}{\partial x_2} = 0$$

Thus

$$(x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 - \mathbf{l})\,\mathbf{C}\,\mathbf{e}_1' = 0$$

and

$$(x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 - \mathbf{l})\,\mathbf{C}\,\mathbf{e}_2' = 0$$

Using the definition of **E**, these two equations may be combined to read

$$(\mathbf{x}\,\mathbf{E} - \mathbf{l})\,\mathbf{C}\,\mathbf{E}' = (0, 0)$$

Re-arranging terms:

$$\mathbf{x}\,(\mathbf{E}\,\mathbf{C}\,\mathbf{E}') = \mathbf{l}\,\mathbf{C}\,\mathbf{E}'$$

As will become apparent later it is useful to define the matrix

$$\mathbf{D} = \mathbf{E}\,\mathbf{C}$$

C is symmetric so $\mathbf{D}' = \mathbf{C}\,\mathbf{E}'$. Assuming we can invert the matrix the solution is therefore:

$$\mathbf{x} = \mathbf{l}\,\mathbf{D}'\,(\mathbf{E}\,\mathbf{D}')^{-1}$$

Since

$$\mathbf{a} = \mathbf{x}\,\mathbf{E} \tag{1}$$

we may also write the solution in the form:

$$\mathbf{a} = \mathbf{l}\,\mathbf{M} \tag{2}$$

where

$$\mathbf{M} = \mathbf{D}'\,(\mathbf{E}\,\mathbf{D}')^{-1}\,\mathbf{E} \tag{3}$$

3.11. These formulae are of general application to the more complicated actuarial models that are discussed later in this paper. For the present we can now insert numerical values to obtain the solution to our particular problem:

$$\mathbf{D} = \begin{pmatrix} 1{\cdot}4363 & 1{\cdot}3177 & 1{\cdot}2088 \\ 1{\cdot}4588 & 1{\cdot}3383 & 1{\cdot}2278 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} {\cdot}0983 & {\cdot}9847 & -{\cdot}0020 \\ {\cdot}0902 & {\cdot}9015 & {\cdot}0002 \\ {\cdot}0812 & -{\cdot}0886 & 1{\cdot}0002 \end{pmatrix}$$

Asset coefficients:

$$\mathbf{x} = (1{\cdot}698 \quad {\cdot}998)$$

Asset cash flows:

$$\mathbf{a} = ({\cdot}270 \quad 1{\cdot}798 \quad {\cdot}998)$$

3.12. The solution in this case is therefore a portfolio comprising 1·698 units of the 2 year stock and ·998 units of the 3 year stock. The cash flows from this portfolio fail to match very precisely the annual liability outgo of 1, but this failure is a result of the limited investments available in the model. Incidentally, the above calculations are very sensitive to rounding errors, and in practice it will be found necessary to carry through the arithmetic with greater precision than the four decimal places shown. As will be shown by later examples, however, solutions tend to be insensitive to the details of the actuarial model.

Direct solution

3.13. Our simple example has been worked out in conjunction with a demonstration of some general formulae. A simpler and more direct method of solution will now be shown. The following table again enumerates all possible cases of future interest rates, this time showing the accumulation of the basic assets $\mathbf{e}_1$ and $\mathbf{e}_2$ and of the liabilities $\mathbf{l}$ with interest to the end of year 3.

Interest rates %		Accumulation of:-		
Year 2	Year 3	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{l}$
8	8	1·1966	1·2246	3·2464
8	10	1·2188	1·2288	3·2880
10	8	1·1988	1·2268	3·2680
10	10	1·2210	1·2310	3·3100

3.14. The mean square ultimate surplus is equal to one quarter of:

$$\begin{aligned}&(1{\cdot}1966\, x_1 + 1{\cdot}2246\, x_2 - 3{\cdot}2464)^2\\ +&(1{\cdot}2188\, x_1 + 1{\cdot}2288\, x_2 - 3{\cdot}2880)^2\\ +&(1{\cdot}1988\, x_1 + 1{\cdot}2268\, x_2 - 3{\cdot}2680)^2\\ +&(1{\cdot}2210\, x_1 + 1{\cdot}2310\, x_2 - 3{\cdot}3100)^2\end{aligned}$$

To minimize this function, take partial derivatives as before and the following equations result:

$$\begin{aligned}5{\cdot}8454\, x_1 + 5{\cdot}9368\, x_2 &= 15{\cdot}8514\\ 5{\cdot}9368\, x_1 + 6{\cdot}0301\, x_2 &= 16{\cdot}0998\end{aligned}$$

The solution to these equations is, as before, very sensitive to rounding errors. If the arithmetic is carried through with maximum precision the answer will be found to agree with that given by the first method. However it should be noted that this more arithmetical approach is practicable only because of the simplicity of the model used for this example.

Remarks

3.15. Although the model used for this worked example is simple the solution illustrates some points which will be found to hold good for the more realistic models discussed later. Firstly, recall the cash flows of the liabilities and the matching portfolio:

At end of year:	1	2	3
Liabilities	1	1	1
Matching portfolio	·270	1·798	·998

3.16. The future interest rate in any year is either 8% or 10% with equal probability, so the mean expected future rate of interest is 9% p.a. At this rate of interest, the discounted present values of both the liabilities and the matching portfolio are equal at 2·531. This is as it should be, although it will be noted that the two concepts of discounting and valuation rate of interest have not been called upon until this point in the discussion.

3.17. With two basic assets to choose from there are infinitely many alternative portfolios which have the same 9% discounted value as the liabilities. What makes the matching portfolio special in relation to the others is that it minimizes the expected ultimate surplus or deficiency; if the rate of interest is not the mean expected rate of 9% (which of course it cannot be in this particular model) then the effect on the outcome is kept to a minimum. This is achieved by arranging the asset cash flows to follow the pattern of the liability cash flows, year for year, as closely as possible. In this way the exposure to uncertain future interest rates is minimized. It will be noted however that this arrangement is a by-product of the matching process, the prime objective of which is to minimize the mean square ultimate surplus. A clearer illustration of this feature is given by an example in §5.19.

3.18. It is interesting to see whether the matching portfolio satisfies the basic immunization conditions. It can be shown that neither this nor any other portfolio taken from this particular two-asset model will immunize the given liabilities at any rate of interest; the assets available are too limited in variety of term to maturity. In any case the immunization conditions would hardly be appropriate here because they are based on a model which assumes a small instantaneous change in the interest rate and no change thereafter. We are allowing in our example for the possibility of quantum jumps in the interest rate on future occasions.

4. THEORETICAL ANALYSIS

4.1. The foregoing example included an algebraic solution to a simple matching problem. It turns out that a theoretical analysis can be developed to provide more of an insight into the subject. Full details of the mathematics are given in a separate paper,[8] the conclusions of which are described next.

Principle of absolute matching

4.2. Let us now regard matching in a generalized sense as the association of a particular portfolio with any given liabilities in order to meet specified criteria. Irrespective of the precise nature of the actuarial model or the criteria, any general theory of matching, if it is to be useful, ought to satisfy two basic conditions:

(i) Whatever the liabilities, the matching should always point to a portfolio which is available from the market model.
(ii) The principle of absolute matching should apply. That is, if the liabilities can be matched absolutely by a portfolio which is available from the market, then the matching should point to that portfolio.

4.3. It is easy to see that the general solution to our worked example meets these fundamental conditions. Taking equations (1) and (2) we have:

$$\mathbf{l}\,\mathbf{M} = \mathbf{x}\,\mathbf{E}$$

which confirms that every liability $\mathbf{l}$ is matched by a portfolio of basic assets in the nominal amounts x_1, x_2 etc. It will be noted that at this stage we have not excluded the possibility that some of these nominal amounts may be negative, a point which is dealt with shortly. Subject to this, condition (i) is satisfied.

4.4. Condition (ii) is verified as follows. If $\mathbf{l}$ can be matched absolutely by a portfolio $\mathbf{x}\,\mathbf{E}$, then

$$\mathbf{l} = \mathbf{x}\,\mathbf{E}$$

because each year's cash flows of assets and liabilities are equal. From equation (3) it will be seen that

$$\mathbf{E}\,\mathbf{M} = \mathbf{E}.$$

Therefore

$$\mathbf{l}\,\mathbf{M} = \mathbf{x}\,\mathbf{E}\,\mathbf{M} = \mathbf{x}\,\mathbf{E}$$

and the transformation **M** meets the condition of absolute matching.

The standard form

4.5. The point of making these observations is that the converse is also true. That is to say that if **M** is *any* matrix transformation which assigns to any liability **l** a portfolio **l M** of assets available in the market (condition (i)) and which satisfies the principle of absolute matching (condition (ii)) then:

$$\mathbf{M} = \mathbf{D}'\,(\mathbf{E}\,\mathbf{D}')^{-1}\,\mathbf{E} \tag{3}$$

for some matrix **D**, where **E** specifies the asset model as before. The demonstration of this fact is not entirely elementary and is dealt with in the associated paper[8].

4.6. It follows that equation (3) not only represents the solution to our worked example; it gives the standard form of any matching transformation. This conclusion is of interest for its generality: we are saying nothing here about the actuarial model or the aspect of the financial process which is to be optimized by matching.

Types of matching

4.7. At this point it is appropriate to review briefly the different types of matching. They are distinguished by the aspect of the financial process which is to be optimized. The worked example was of an unconstrained match under a stochastic model, in which the target for optimization is E_2, the mean square ultimate surplus. The unbiased match was defined earlier in terms of portfolios which minimize E_2 subject to the constraint of the mean ultimate surplus being zero. The positive match minimizes E_2 subject to all nominal asset holdings being non-negative. Other targets for optimization can be defined under stochastic or deterministic models, but they will not be considered here.

The positive match

4.8. Of the various types of matching, the positive match stands out as being appropriate for modelling most realistic situations. Given an appropriate actuarial model it can be shown that there is a unique positive match to any liability, thus confirming the importance of this concept. In broad terms the standard form of equation (3) applies, although it is first necessary to determine which of the basic assets available in the market are correct for the given liabilities. To illustrate this point, the example in § 5.6 shows that an unconstrained match to a deferred annuity involves negative holdings of short-dated assets. For a positive match such assets (though not necessarily the same ones) must be excluded.

4.9. The selection of the appropriate subset of basic assets for any particular match is not straightforward. This problem belongs to the subject of quadratic programming—the optimization of a quadratic function subject to linear constraints—and it bears a close resemblance to the portfolio selection problem described by Markowitz[9]. Markowitz gave an algorithm for the solution of his problem, and other general algorithms for quadratic programming have been published. Quadratic programming was also described by Wegner[10].

4.10. However the evaluation of a positive match is a particular problem for which the more general quadratic programming methods do not seem particularly well suited. A special algorithm for the positive match has been developed and used to obtain all the results in this paper. It appears to be efficient and to minimize the potential problems of rounding errors. Details are given in the associated mathematical paper[8].

Principle of invariance

4.11. The standard form for a matching transformation (equation (3)) involves reference only to the two matrices **E** and **D**. It will be recalled that **E** contains information relating only to future cash flows from existing investments. Therefore all the other information required for the matching, namely the actuarial model of future conditions and the target for optimization, is contained in matrix **D**. This matrix calls for inspection.

4.12. It will be noted that **D** and **E** bear a symmetrical, or dual relationship in equation (3). Both matrices are of the same dimensions. Referring back to the worked example, **E** comprised two row vectors which defined the basic assets, and **D** also comprised two row vectors:

$$\mathbf{d}_1 = (1{\cdot}4363 \quad 1{\cdot}3177 \quad 1{\cdot}2088)$$
$$\mathbf{d}_2 = (1{\cdot}4588 \quad 1{\cdot}3383 \quad 1{\cdot}2278)$$

The liability cash flow vector was:

$$\mathbf{l} = (1 \quad 1 \quad 1)$$

and the matching portfolio cash flow vector was:

$$\mathbf{a} = ({\cdot}270 \quad 1{\cdot}798 \quad {\cdot}998)$$

It can be seen that the following equalities hold between scalar products:

$$\mathbf{l}\,\mathbf{d}_1' = \mathbf{a}\,\mathbf{d}_1' = 3{\cdot}963$$
$$\mathbf{l}\,\mathbf{d}_2' = \mathbf{a}\,\mathbf{d}_2' = 4{\cdot}025$$

4.13. These equalities are exact, they are not coincidental and they do not depend upon the particular way in which **D** was defined in the worked example. It is a fact that whatever the matching transformation, if **a** is the match to **l** then

$$\mathbf{l}\,\mathbf{d}' = \mathbf{a}\,\mathbf{d}'$$

for every row $\mathbf{d}$ in $\mathbf{D}$. This is true for any $\mathbf{l}$ and corresponding match $\mathbf{a}$. The proof is simple:

From equation (3)	$\mathbf{D}'$	$= \mathbf{M}\,\mathbf{D}'$
So for any column $\mathbf{d}'$	$\mathbf{d}'$	$= \mathbf{M}\,\mathbf{d}'$
Therefore for any $\mathbf{l}$	$\mathbf{l}\,\mathbf{d}'$	$= \mathbf{l}\,\mathbf{M}\,\mathbf{d}' = \mathbf{a}\,\mathbf{d}'$

4.14. In this sense it can be said that the scalar product of a cash flow vector with any of the vectors $\mathbf{d}$ is invariant under the matching transformation. The row vectors of $\mathbf{D}$ will therefore be referred to as invariants. We can state a principle of invariance. In relation to any matching of liabilities with portfolios selected from a number of distinct basic assets, there is an equal number of distinct vectors $\mathbf{d}$ with respect to which the scalar products $\mathbf{l}\,\mathbf{d}'$ are invariant as above.

4.15. It was said above that the matrices $\mathbf{D}$ and $\mathbf{E}$ bear a dual relationship to each other. So therefore do the invariants and the basic assets. They are counterparts, equal in number, and the principle of invariance is mathematically a counterpart to the principle of absolute matching. These rather abstract conclusions find an application in the discussion of asset valuation. (See § 6.36 onwards).

Discounted present value

4.16. If v is given the conventional compound interest meaning, define the vector:

$$\mathbf{v} = (v, v^2, v^3, \ldots, v^m)$$

where m is the number of cash flows in vector $\mathbf{l}$. Then the discounted present value of $\mathbf{l}$ at the given rate of interest is the scalar product $\mathbf{l}\,\mathbf{v}'$. If $\mathbf{a}$ represents the cash flows from any portfolio of assets then its present value is $\mathbf{a}\,\mathbf{v}'$. If the present values of the liabilities and assets are equal then:

$$\mathbf{l}\,\mathbf{v}' = \mathbf{a}\,\mathbf{v}'$$

4.17. The conventional actuarial approach is to judge the position of a fund by this equation at one or more suitable rates of interest. If this is to be the crucial test then, in our theory of matching, $\mathbf{v}$ would assume the role of an invariant in the matching of assets to liabilities. It would appear as one of the rows in matrix $\mathbf{D}$.

4.18. In fact, looking at the ratios of successive elements in the invariants $\mathbf{d}_1$ and $\mathbf{d}_2$ of the worked example (see § 4.12) it will be seen that both these invariants are very similar, to within a scaling factor, to $\mathbf{v}$ at the mean interest rate of 9%. Although the similarity is close, neither is identical. This demonstrates that although the discounted present value can assume the role of an invariant, as in the conventional equation of value, this is not an automatic consequence of matching under a stochastic model of the interest rate.

Inflation

4.19. It will be apparent from the discussion of matching for pension funds that a stochastic model is required for inflation as well as interest. Such an extended actuarial model can then be used to represent not only the mean expected rates of such factors as the growth of pay levels and of dividends on equity investments, but also the uncertainty in these factors and the correlation between them. This is not a point of difficulty because at no stage in the discussion have we had to call into play any aspects of the actuarial model. General conclusions such as the existence of a unique positive match remain unaffected by the introduction of inflation in the model. Equally there is no change in the algorithms to derive solutions.

4.20. The way in which inflation enters the stochastic model can be illustrated by reference to the worked example in which the interest rate in any year is either 8% or 10% with equal probability. For an equally simple model of inflation let the rate in any year, as measured by a suitable index, be either 6% or 7% with equal probability. We need to model the effects of cash flows upon the ultimate surplus, for which purpose inflation up to the time of the cash flow must be represented. Mathematically the accumulation of unit cash flow from time t to time 3, namely F_t, depends upon the effect of inflation over the period between time 0 and time t.

4.21. For example F_2, which took the values 1·08 or 1·10 in the worked example, could now result in any one of eight possible outcomes with equal probability, as shown in the table:

Inflation Year 1 (1)	Inflation Year 2 (2)	Interest Year 3 (3)	F_2
1·06 or 1·07	1·06 or 1·07	1·08 or 1·10	(1) × (2) × (3)

4.22. It would be possible to re-work the earlier example on the footing that all asset and liability cash flows depend in their amount on the foregoing inflation index up to the date of payment. This would be done by replacing the previous values of F_t with those calculated to include the stochastic effect of inflation as above. In a practical case it may also be necessary to model cash flows which depend on inflation during only part of the period prior to payment. An example would be a payment of pension from a final salary scheme in which the amount of benefit is fixed as from the date of retirement. This model gives rise to distinct values of F_t according to the extent to which inflation affects the cash flows. Taking the case of F_2 above we can envisage the following types:

(i) Cash flows fully indexed to inflation: eight outcomes of (1) × (2) × (3)
(ii) Cash flows indexed to inflation in year 1 only: four outcomes of (1) × (3).
(iii) Cash flows independent of inflation: two outcomes of (3).

4.23. Clearly this model is more complicated than that without inflation. In particular matrix **C**, which is derived from F_t, needs to be enlarged in the number of rows and columns to distinguish not only cash flows occurring at different times but also inflation operating over different periods. Further details are given in the associated paper(8). This complication adds considerably to the computing requirements to obtain a solution, but there is no difference in principle between the calculations which need to be made to find the match in our simple worked example and those required to find the match to a realistic pension fund model.

Uniqueness

4.24. The uniqueness of a match follows from the fact that the principle of invariance (§ 4.14) imposes the same number of constraints on the solution as there are degrees of freedom in the choice of portfolio, namely the number of basic assets. However the form of the matching transformation relies on the assumption that matrix $\mathbf{E}\,\mathbf{D}'$ can be inverted (see equation (3)). It turns out that two conditions must be satisfied to ensure a unique match, one concerning the asset model and one concerning the stochastic model of interest and inflation.

4.25. The condition on the asset model is that the base assets are linearly independent. What this means in practice is, for example, that asset number 3 is not exactly equivalent to twice asset 1, nor for example that its cash flows are identical to the sum of those from assets 1 and 2. This condition is not particularly onerous, because if it is found that asset 3 is linearly dependent on other base assets then all that is necessary is to remove it from the set of base assets. Portfolios selected from the residual set will still be capable of generating the cash flows of any portfolio which included asset 3, but the potential ambiguity in the selection of base assets by reference to their cash flows will have been removed. An example of a linearly independent set of base assets is a range of fixed interest stocks all of which are due to be redeemed on different dates.

4.26. The condition on the stochastic model is that the model should be 'fully stochastic'. The precise meaning of this condition is given in the associated paper(8), but some examples will help to explain it. An actuarial model of interest and inflation is *not* fully stochastic if either:

(a) the interest rate in any one future year is known with certainty;
(b) the inflation rate in any one future year is known with certainty;
(c) the difference between the interest and inflation rates in any one future year is fixed and known with certainty.

4.27. The reasons why these cases would not necessarily give unique matches are fairly easy to see. For example suppose the interest rate in year 2 could only be a fixed 10% (case (a)). Then the outcome of a cash flow of 1 at the end of year 1 will always be identical to that of a cash flow of 1·10 at the end of year 2. This being so, the matching requirements may, depending on the basic assets, be met by more than one, indeed by an infinite number of alternative portfolios which differ only in their arrangement of cash flows at these two dates.

4.28. Case (c) is also worth examining, because it will be referred to in later discussion. Suppose that the interest and inflation rates both vary year by year in accordance with a statistical model but that in year 2 the two rates differ by a constant 2%. (Strictly let $(1+i)/(1+e) = 1{\cdot}02$ be constant where i is the rate of interest and e is the rate of inflation). Consider cash flows which are dependent in their amount on inflation over the period up to the date of payment. Then the outcome of a cash flow of 1 plus inflation at the end of year 1 will always be identical to that of a cash flow of 1·02 plus inflation at the end of year 2, and the situation which results is similar to that of case (a).

4.29. In practice, if we are representing uncertain economic conditions in an actuarial model it is unlikely that we shall feel inclined to introduce a determinate element into the stochastic model. However case (c) needs to be borne in mind because of the tendency to think of the 'real rate of return' as a stable, if not fixed parameter in pension fund finance. Subject to this, any reasonable stochastic model is likely to satisfy this condition for uniqueness of match.

5. EXAMPLES

5.1. This section presents the results of applying the theory to some simple cases using differing actuarial models. Some further ideas are also introduced.

Interest models

5.2. In actuarial work 'interest' is a simple word with a complicated meaning. We must now construct stochastic models of the interest rate, so it is necessary to define the meaning for this purpose. 'Interest' means the overall return arising from the investment of future income followed by eventual disinvestment. The overall return comprises true interest, dividends, rents and capital losses or gains. Ignoring any question of taxation in this discussion, income and capital need not be distinguished for this purpose. If future borrowings are required to cover liability cash flows, interest is also understood to mean the overall cost of borrowing. We shall assume that borrowing and investment rates are similar at any time.

5.3. At future times when a fund is not absolutely matched and has a net income for investment, the interest earned on such future net investment will depend upon financial conditions then prevailing. We shall be making assumptions relating not to the details of future investment markets, just 'the rate of interest' to be obtained at future times. Two types of future investment will be distinguished:

(a) interest-bearing short term deposits;
(b) longer dated investments in which the redemption yield prevailing at the date of investment is secured for the subsequent period until eventual realization for cash

5.4. Our models of the interest rate must incorporate the element of uncertainty by assigning probabilities, or a probability distribution, to future rates of interest. The worked example used the simplest conceivable stochastic model of two rates of interest with equal probabilities. This model will be used along with three others, as detailed below, for demonstrating the nature of the basic results.

Model A: Future net investment placed on deposit, each year's rate being 8% or 10% with equal probability and being independent of rates in other years.

Model B: Future net investment is long term to eventual realization. The interest rate changes every year, starting at 9% in the first year and with independently distributed annual changes thereafter. This model of the interest rate is therefore a random walk. The probability distribution of the interest rate changes is log-normal (so that changes in the force of interest are normally distributed) with mean zero and standard deviation ·5%.

Model C: As model B but starting at 7% instead of 9%.

Model D: As model B but starting at 11% and with a tendency to drift to lower interest rates from the initial 11%. The mean drift is ·5% for each annual change in the interest rate. This model will be considered only in relation to a 10 year term of assets and liabilities so no barrier will be imposed to the downward movement of the interest rate.

The formulae for calculating matrix **C** for the various stochastic models are given in the associated paper.[8]

Annuities certain

5.5. First consider the matching of annuities certain by fixed interest stocks. Let us postulate a range of hypothetical stocks with a coupon rate of 10% payable annually in arrears and terms to redemption ranging from 1 to 10 years. We thus have ten basic assets in the model, which are conveniently identified by their outstanding term. For example, using the established notation:

$$\mathbf{e}_3 = (\cdot 1 \quad \cdot 1 \quad 1{\cdot}1 \quad 0 \ldots \quad \ldots 0)$$

For the model of liabilities let all cash flows also occur annually in arrears. If the liability is an immediate annuity certain of unit amount for ten years, i.e.:

$$\mathbf{l} = (1 \quad 1 \quad 1 \quad \ldots \quad \ldots 1)$$

then we would expect to be able to find an absolute match from the available assets. This is confirmed by calculating the positive match for **l**; for all four interest models A, B, C and D the positive match is the same, namely the portfolio with the following composition.

Match for liability 1

Asset number	Nominal holding
1	·386
2	·424
3	·467
4	·513
5	·564
6	·621
7	·683
8	·751
9	·826
10	·909

This portfolio does indeed generate unit cash flow in each of the 10 years.

5.6. Next consider liability 2, an annuity certain of 5 years deferred 5 years. The unconstrained match and the positive match differ for this deferred annuity, and they are both shown below for interest model A.

Match for liability 2—interest model A

	Nominal holding:	
Asset number	Unconstrained match	Positive match
1	−·235	0
2	−·259	0
3	−·285	0
4	−·313	0
5	−·345	0
6	·621	0
7	·683	0
8	·751	·630
9	·826	·825
10	·909	·929

5.7. It may be verified that the unconstrained match is absolute. The positive match excludes the first seven assets and is not absolute because the interest received during the period of deferment must be reinvested. The positive match therefore depends upon the interest model, and the results for the four alternative interest models described above are shown in the following table. These and all other results have been calculated exactly using the method referred to in § 4.10.

Positive match for liability 2—nominal holdings

	Interest model:			
Asset number	A	B	C	D
1–6	0	0	0	0
7	0	·219	·227	·196
8	·630	·784	·778	·785
9	·825	·859	·861	·716
10	·929	·526	·589	·618

5.8. It can be observed that the four portfolios are somewhat similar in their broad composition. Although there are marked differences between the four interest models, including differences in the mean future rate of interest, these factors are less significant than the need to avoid assets which mature during the deferment period and the need to match closely the pattern of liability cash flow thereafter.

Market values

5.9. Assuming that the base assets have well-defined market values on any particular date, the market value of the positive match portfolio on that date may be determined. This market value will be referred to by the abbreviation M.V. It will be appreciated that M.V. is a function of the liabilities and the actuarial model as well as market conditions generally at the relevant date. However it is independent of the actual assets held in the fund.

5.10. Although M.V. can in principle be evaluated as at any number of dates, a convention will be adopted in this paper that the valuation date to be used will be that occurring half way through the first time step. Therefore as viewed from the valuation date, cash flows represented in the model occur at times $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$ etc. This convention will be useful in later applications where cash flows during a period of 1 or 5 years are treated as if they occur at the mid-point of the period.

5.11. Let us postulate two alternative market conditions. The first of these reflects a level yield curve with all assets showing a redemption yield of 11% p.a. The second market reflects a yield curve which declines quite steeply with term. Details are as shown in the table:

	Market (i)	Market (ii)	
Asset number/term	Market values	Redemption yield (% p.a.)	Market values
1	1·044	12·0	1·039
2	1·036	11·8	1·025
3	1·028	11·6	1·016
4	1·021	11·4	1·010
5	1·015	11·2	1·008
6	1·009	11·0	1·009
7	1·004	10·8	1·013
8	·999	10·6	1·019
9	·995	10·4	1·027
10	·992	10·2	1·037

(Note that the current redemption yields are somewhat out of line with the three models of the future interest rate, other than model D. In effect therefore models A, B and C are predicting an imminent fall in the current rate.)

5.12. We can readily determine M.V. for the cases which have already been considered and for each of these two markets. The values are as follows.

Liability	Interest model	Market (i)	Market (ii)
1 (immediate)	any	6·205	6·272
2 (deferred)	A	2·372	2·453
	B	2·379	2·448
	C	2·445	2·517
	D	2·307	2·375

Notice that the values of M.V. under models A and B are very close, despite the differences in the composition of the matching portfolios. The two models imply the same mean future rate of interest, so the indication is that M.V. may be rather insensitive to the precise distribution of the interest rate about the mean.

Matching rate of interest

5.13. Two distinct types of interest rate have already been introduced into the discussion: the future rate which follows a statistical pattern of behaviour and the present market rate which is known at the valuation date but which can depend on term. To these can be added a third type, the rate at which the discounted value of the liabilities equals M.V. This will be called the matching rate of interest. The rates corresponding to the above table of M.V. are as follows.

Liability	Interest model	Market (i) (% p.a.)	Market (ii) (% p.a.)
1 (immediate)	any	11·0	10·7
2 (deferred)	A	10·6	10·1
	B	10·6	10·1
	C	10·1	9·7
	D	11·0	10·6

5.14. It will be noted that the matching rate for liability 1 under market (i) is the same as the market rate of interest. This is an automatic consequence of the absolute matching. The corresponding rate under market (ii) is an average of the market redemption yields for terms corresponding to the liabilities, weighted towards the longer terms in the 10 year range.

5.15. The matching rates for liability 2, the deferred annuity, depend on the interest model. They are mostly below current market rates because all four models are predicting lower interest rates in the future. There is little to choose between the results for models A and B, both of which are predicting a mean future interest rate of 9%. Model C is predicting a still lower rate of 7%. Model D is also predicting a fall in the interest rate, but more gradually over a period of years. The extent to which the matching rate differs from the current market rate is a measure of the predicted mean future rate of interest and of the degree of mis-matching, which in turn depends on the deferment period. This is demonstrated clearly in the following table which shows the matching rate under model C for a range of immediate and deferred annuities of unit amount.

Period of deferment (Years)	Period of payment (Years)	Market (i) (% p.a.)	Market (ii) (% p.a.)
0	10	11·0	10·7
2	8	10·5	10·2
4	6	10·2	9·9
6	4	10·1	9·6
8	2	9·8	9·3

5.16. A further observation on these results is that the matching rate under market (ii) tends to be lower than that under market (i). The difference widens a little for the longer terms of deferment, reflecting the shape of the yield curve. For the rest of the discussion on matching by fixed interest stocks we shall suppose that market (i) with its level yield curve describes the position at the valuation date. This assumption is arbitrary and none of the following arguments depend upon it.

Mean and standard deviation of surplus

5.17. The definition of the positive match is that it minimizes E_2, the mean square ultimate surplus. A direct measure of the degree of mis-matching is therefore the minimum value of E_2 which is attained by the matching portfolio. Since the positive match is not constrained to ensure that the mean ultimate surplus (denoted E_1) is zero, it is also relevant to know the value of E_1. From these statistics can be calculated the standard deviation of the ultimate surplus arising from the matching portfolio. This is an interesting statistic, but not as interesting as its present value counterpart. We shall therefore define a new statistic (S.D.) which is the standard deviation of ultimate surplus from the matching portfolio discounted at the matching rate of interest. Mathematically:

$$\text{S.D.} = \frac{\sqrt{E_2 - E_1^2}}{(1+i)^{m-\frac{1}{2}}}$$

5.18. The values of E_1, E_2 and S.D. for liabilities 1 and 2 (and 11% market) are as follows.

Liability	Interest model	E_1	E_2	S.D.
1 (immediate)	any	0	0	0
2 (deferred)	A	−·0002	·00107	·013
	B	−·0008	·00456	·026
	C	−·0006	·00366	·024
	D	−·0008	·00475	·026

The values of E_1 in these examples are negligible and the same is true of all other cases mentioned in this paper. This experience suggests that there would be little point in re-defining the match either to minimize variation of surplus about the mean or to minimize E_2 subject to E_1 being zero.

Missing assets

5.19. Suppose that some of the assets which were postulated in the foregoing examples are excluded from the market model, representing a situation in which not all cash flows can be matched by simultaneous redemptions of available fixed interest stock. The definition of a positive match does not call for a complete set of base assets and we can re-work the previous examples with a reduced set. For example, reducing the number of base assets from ten to five, retaining those which mature after 1, 3, 5, 7 and 9 years, produces the following results for model C.

Nominal holdings in matching portfolios
– restricted asset set, model C

Asset number/term	Liability 1	Liability 2
1	·553	0
3	·923	0
5	1·106	0
7	1·164	·356
9	2·397	2·098

Cash flows from matching portfolios

Year	Liability 1	Liability 2
1	1·168	·245
2	·559	·245
3	1·482	·245
4	·467	·245
5	1·572	·245
6	·356	·245
7	1·520	·602
8	·240	·210
9	2·636	2·308
10	0	0

5.20. There is no longer an absolute match to liability 1, but it can be said that the matching process 'does its best' to minimize the exposure to future changes in the interest rate. The cash flow from the matching portfolio alternates about the fixed annual liability outgo of 1, so that net investment during the 10 years is kept to a minimum and followed by disinvestment as quickly as possible. This empirical property of the matching process, which is confirmed by other examples, gives reason for expecting the nature of a matching portfolio to be relatively insensitive to the details of the interest model.

5.21. Statistics of the above matching portfolios are as follows:

	Liability 1	Liability 2
M.V.	6·202	2·446
Matching rate of interest (%)	11·0	10·1
S.D.	·012	·024

Comparing these statistics with the earlier results for a full set of assets (model C, market (i)) shows that the results are almost identical with the exception of S.D. for liability 1, which is no longer zero because the restricted assets cannot match this liability absolutely.

Valuation of liabilities

5.22. The valuation of liabilities is a familiar concept, yet an imprecise one. Valuation by discounted cash flow is just a technique to answer the question "what fund is required now to meet these liabilities?" This question begs others, in particular what exactly is meant by meeting the liabilities when future conditions are uncertain. In practice there can be no uniquely correct answers to such questions; similar liabilities can be valued at differing amounts according to a variety of circumstances.

5.23. The calculation of a matching portfolio may be thought of as another technique of valuation. The result of the calculation is M.V., which may be compared directly with the market value of the fund. As indicated by the examples, valuation of annuities by M.V. produces results which appear reasonable in the light of the assumptions made. In particular an immediate annuity is valued at the current market rate of interest even if the supply of fixed interest stocks does not cover all possible redemption dates and the interest rate is expected to fall significantly. Given a cautious view of future interest rates, the effective (matching) rate of interest for valuation of deferred annuities decreases with increasing term of deferment.

5.24. It might be argued that a valuation by matching is inappropriate because if the result is a mean ultimate surplus (E_1) of about zero, then there will be a roughly 50% chance of a deficiency. If this is a concern then the solution is already at hand. Consider the last example where the immediate annuity of liability 1 would be valued at M.V. $=6{\cdot}202$. Taking the probability distribution of the ultimate surplus as approximately normal, we can arrange a $97\frac{1}{2}\%$ probability of surplus by increasing E_1 by two standard deviations. The discounted value of E_1 itself is trivial, so we need only increase M.V. by $2\times$ S.D., to a revised figure of:

$$6{\cdot}202+(2\times{\cdot}012) = 6{\cdot}226$$

If the liabilities are valued at 6·226 and the fund is of precisely this amount and invested in the asset proportions of the matching portfolio then there will be a $97\frac{1}{2}\%$ probability of surplus on the valuation assumptions. The adjustment in this case is relatively small because liability 1 is a well matched immediate annuity. The adjustment would be greater for the deferred annuity, liability 2, for which the value S.D. as shown in § 5.21 is proportionately much larger.

5.25. Seen in this light, matching is a technique of valuation to arrange any specified probability of ultimate surplus on the basis of the given assumptions. This, it is suggested, provides a respectable answer to the questions posed in § 5.22. The idea of a matching valuation is not a new one; it was discussed by

Benjamin[11] in the context of applying games theory to the selection of an optimal portfolio for a life office. His model used detailed assumptions about the precise terms of future investment and disinvestment rather than about their probabilities.

Index-linked annuities

5.26. The last example in this section concerns the matching of index-linked annuities by index-linked stock. As from the date of purchase of any such stock prospective payments of interest and of capital on redemption are increased in line with inflation, as measured by the Retail Price Index (R.P.I). For this purpose the small time-lag in the indexing of Government stocks will be ignored. Consider annuities which are increased by the same index. If such an annuity were immediate and if the range of available index-linked stocks were sufficiently extensive in term to maturity date, then we should be able to arrange an absolute match in the generalized sense of zero variation in the ultimate surplus. In fact the index-linked stocks currently available in the U.K. are still somewhat sparse in their redemption dates, and absolute matching is not possible. It is therefore of interest to apply the matching procedure described in the preceding paragraphs with a view to valuing index-linked annuities.

5.27. The way in which inflation is brought into the stochastic model was described in §§4.19 to 4.23. We shall use the following model for the present purpose.

Model E: Interest—future net investment placed on deposit. The deposit rate starts at 9% in the first year and then performs a random walk with independent annual changes. The probability distribution of the changes is log-normal with mean zero and standard deviation ·5%.

Inflation—the rate each year falls short of the deposit rate of interest by the 'real rate of return', which is independently distributed, log-normal, with mean 3% and standard deviation 1%. The mean future rate of inflation is 5·82% (since 1·09/1·03 = 1·0582). The inflation rate therefore performs a random walk which is correlated with the interest rate.

5.28. The available investments for this matching will be taken as a notional range of six index-linked stocks, all of which have a coupon of $2\frac{1}{2}$% payable annually in arrears and with terms to redemption of 5, 10, 15, 20, 25 and 30 years from the valuation date. All six stocks are priced at the valuation date to produce a real redemption yield of $2\frac{3}{4}$% relative to inflation. (This market is somewhat idealized in order to simplify the example and help illustrate some points; it would be easy to replace this notional range of stocks by those actually available on any chosen date.)

5.29. The liabilities consist of a monthly index-linked single life annuity to a male who attains age 75 at the valuation date. The mortality is PA(90) and the liabilities will be taken as zero after 30 years. (It will be recalled from § 2.10 that we treat mortality as a determinate factor in the conventional manner.) The matching portfolio is as follows:

Asset term	Nominal holding
5	4·644
10	1·769
15	·881
20	·276
25	·050
30	·009

The relevant statistics are:

M.V.	7·599
Matching rate of interest (%)	8·70
S.D.	·049

5.30. We can also define another statistic, the matching real rate of return at which the liabilities should be valued net of inflation to produce M.V. We have:

$$\frac{1+\text{matching rate of interest}}{1+\text{mean rate of inflation}} = \frac{1{\cdot}0870}{1{\cdot}0582} = 1{\cdot}0272$$

Therefore the matching real rate of return is 2·72%. This rate is fractionally lower than the current real redemption yield of 2·75%. The small difference can be attributed to the cost of having to borrow in future years at a real rate of 3% p.a. on average in order to finance annuity payments prior to redemptions of the stocks. Following § 5·24, the annuity should be priced at M.V. $+(2\times\text{S.D.})=7{\cdot}697$ in order to arrange a $97\frac{1}{2}\%$ probability of surplus. At this value the effective valuation rate of interest relative to inflation is nearer 2·5%.

5.31. If the above matching is repeated with one change in the model, namely a reduction in the standard deviation of the real rate of return from 1% to ·1%, the resulting portfolio and valuation are virtually identical. The only significant difference is in the value of S.D., which not surprisingly is reduced. This demonstrates that the matching portfolio is fairly insensitive to the variation of the real rate of return about its mean expected value and that only a slight degree of uncertainty in this parameter is required to yield a unique match with a sensible term structure. However it should be noted that if the real rate of return is a fixed 3%, with no degree of uncertainty, then there is no unique matching portfolio. This bears out the general point made is § 4.28 about a model which is not fully stochastic.

6. APPLICATIONS TO PENSION FUNDS

6.1. The ideas which have been discussed in this paper are now collected together in the form of applications to various problems of pension finance.

Actuarial model

6.2. The following model will be used.

Model F: Interest—future net investment is long term to eventual realization. The interest rate changes each year, each year's rate being independently distributed, log-normal, with mean 9% and standard deviation 1% p.a.

Inflation—the rate as measured by the R.P.I. each year is independently distributed, log-normal, with mean 6% and standard deviation 1%, and is uncorrelated with the interest rate.

Increases of earnings—the rate each year is precisely $1\frac{1}{2}\%$ more than the increase in R.P.I., and thus averages $7\frac{1}{2}\%$ p.a. There is no additional provision for age-related increments.

Pension increases—all pensions in payment increase at a fixed 3% p.a., in accordance with the rules of the scheme.

Dividend growth—the rate of growth of equity dividends is precisely $1\frac{1}{2}\%$ p.a. less than the increase in R.P.I., and thus averages $4\frac{1}{2}\%$ p.a.

6.3. In conventional terminology this actuarial model is equivalent to a basis of 9% p.a. interest, $7\frac{1}{2}\%$ p.a. salary increases, 3% pension increases and $4\frac{1}{2}\%$ p.a. dividend growth. However this conventional-style basis has been extended by the inclusion of stochastic effects in all parameters other than pension increases. It should be noted that this model is fully stochastic, in the sense described in § 4.26. This is true despite the assumption of fixed differentials between the growth rates of R.P.I., earnings and dividends. It would be possible to construct a more complicated and more realistic model in which these differentials are also variable, but the model is being kept as simple as possible for present purposes and it seems unlikely that a more complex model would yield results significantly different from those described below.

6.4. For the model of the present investment market we shall assume that all holdings of shares are in the proportions underlying the Financial Times–Actuaries All-Share index, and that the market price of the index fund increases in proportion to dividend growth. In other words future fluctuations in the All-Share index dividend yield will be ignored. The valuation date will be taken as 30 September 1983, when the dividend yield index was 4·80%. Dividends will be assumed to be paid continuously throughout the year. In effect our asset model comprises a range of equity-index assets, because we can assume sale of the investment at the end of any year on market terms as specified above. For example over an 80 year lifetime of the fund we would have a basis of 80 equity-index assets, one such asset being realized at the end of each year.

6.5. In addition to the equity investments the model will include a range of dated fixed-interest stocks. For ease of discussion an idealized set of stocks will be assumed. All have a coupon of 10% payable annually in arrears and they mature at the end of years 3, 8, 13, 18, 23 and 28. Recalling our convention that the valuation date is half-way through the first time-step, the first stock matures after $2\frac{1}{2}$ years at 31 March 1986, and a full year's interest will be paid each March. All fixed interest stocks are priced at the valuation date to produce a redemption yield of 11% p.a.

The liabilities

6.6. Various calculations, which will not be described here in detail, confirm that if the liabilities are completely fixed in monetary terms, the matching portfolio is selected from the fixed interest stocks above. If the liabilities are fully indexed with inflation the matching portfolio is selected from the share index fund. There is no difficulty in selecting the correct type of portfolio from the mixed asset model in those black and white situations. Matters become more interesting when the liabilities themselves are also mixed between the inflation-linked and monetary varieties.

6.7. It is often said that the liabilities of a final salary pension scheme are mostly if not wholly linked to inflation and that holdings of fixed interest dated investments are justified only on grounds of providing income, security and diversification. This may be true of some schemes but it should not be accepted as a generality. The full inflation linking usually operates in respect of members only so long as they remain in service, and if their final earnings are averaged for pension purposes the linkage is effectively broken before the leaving date.

6.8. Many schemes do not increase the preserved benefits of early leavers during the deferment period; legislation may force general changes in this respect but the resulting statutory increases of preserved benefits will probably continue to be ascertainable in monetary terms. Many schemes are contracted out of the State earnings-related scheme and do not increase Guaranteed Minimum Pensions (GMPs) of members when in payment. GMPs represent a growing proportion of the accrued liabilities of such schemes.

6.9. Many schemes are funded without regard for the cost of possible future increases of pensions in payment beyond those if any which are prescribed by the rules. Such a funding policy need not directly affect the extent to which additional discretionary increases may be awarded, but it does imply that from the point of view of the actuary the pensions are of the monetary kind when in payment and additional finance will be required to cover any future discretionary increases.

6.10. Our model scheme prescribes a fixed 3% annual increase on the total pension and model F does not admit the possibility that any greater increases will be awarded. For present purposes it is unnecessary to know whether discretionary increases will be awarded in addition; suffice it to note that if the funding policy has been agreed in this form then it would be proper for the actuary to regard the pensions in payment as purely monetary liabilities. Furthermore the bulk of liabilities to members approaching retirement would be undergoing a metamorphosis from the inflation-linked to the monetary kind.

A simplified demographic model

6.11. Pension fund calculations are usually complex, especially when several demographic factors are involved. In order to avoid compounding the complications with the matching procedure, the first example uses a prototype model of a final salary scheme with a 30 year timespan and no demographic factors. The details are as follows:

Minimum entry age	45
Retirement age	65
Probability of reaching retirement age	100%
Term of pension payments	10 years certain

Distribution of pension liabilities (in appropriate units) as below:

	Age next birthday	Annual pension	
	46–50	40	
	51–55	50	
	56–60	60	
	61–65	60	
Total in service:		210	(prospective amounts ignoring future inflation)
	66–70	55	
	71–75	50	
Total in retirement:		105	(current amounts)
Grand total:		315	

The distribution of pensions within each 5 year age group is one fifth at each age. Pensions are effectively paid continuously, and for the membership as a whole the dates of pay and pension increases and of retirements are also effectively continuous throughout each year. For the first set of calculations future contributions are ignored.

6.12. The calculation of a match under model F has been made in two ways: first using annual time-steps and secondly using 5 yearly steps in which all cash flows are aggregated to the mid-points of the periods. Relative to the valuation date the cash flows therefore occur at times $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$ etc. for the first calculation and at times $2\frac{1}{2}$, $7\frac{1}{2}$, $12\frac{1}{2}$ etc. for the second. The model of fixed interest stocks has been altered slightly for the 5 year grouping by adopting a new set of notional stocks which have a coupon of 50% payable every 5 years in arrears, coincident with the other 5 yearly cash flows. The results are tabulated below.

Prototype model F—ignoring future contributions

Assets	Redemption dates	Nominal holdings derived from: 1 year steps	5 year steps
Equity index	1–5	55	177
	6–10	163	171
	11–15	319	314
	16–20	421	420
	21–25	213	127
	26–30	0	0
Fixed interest	3	0	0
	8	0	0
	13	0	0
	18	0	0
	23	186	512
	28	364	0
M.V.		1,702	1,706
Matching rate of interest (%)		9·9	9·9
S.D.		3·8	3·1

6.13. The statistics with which we shall be concerned in pension fund applications are M.V. and the matching rate of interest. The similarity of these two statistics for the 1 year and 5 year time-steps will be taken as justification for using 5 year steps in the remainder of this paper. This eases the computing requirements. There are seen to be differences between the two sets of results in the distribution of fixed interest investments by term, and similarly in the timing of sales of equities. Such differences may be expected in view of the approximations enforced by a 5-year grouping of projected cash flows, and they are considered to be of relatively minor significance.

Matching by term

6.14. The immediate observation to be made upon the matching portfolio is that the distribution of equity assets, according to the various periods to realization, broadly corresponds to the terms of the liabilities. This observation appears a natural one, but it contrasts with views expressed by Fellows[(6)]. Fellows considered that the merits of equity investment for pension funds lay in the matching of 'real assets' to 'real liabilities' but not in the indefinite length of their term. He argued that if the real rate of return is constant then the appropriate asset term would be zero, implying dead short investment.

6.15. In the discussion on Fellows' paper, some speakers took issue with this argument and considered that the logic relied too heavily on the initial premise of a constant real rate of return. The analysis described in this paper supports that objection. In the terms of § 4.26, Fellows' model is not fully stochastic and any term of assets would do equally well. However as soon as the factor of uncertainty is acknowledged in relation to the differential between interest and inflation rates, matching considerations dictate a specific set of 'real assets' according to term. This was also illustrated in § 5.31, where the example concerned an index-linked annuity matched by index-linked stocks.

6.16. There remains the point that our investment model assumes an unchanging dividend yield and ignores fluctuations in market values on sale. However now that we have dated index-linked stocks to substitute for equities if desired, this objection to our own model does not seem material. Matching by term is appropriate for real liabilities.

6.17. Turning to the fixed interest assets in the matching portfolio it appears at first sight that matching by term does not hold good in this example. We have some pensions in payment which are increasing at a fixed 3% p.a. and which could be closely matched by a holding of fixed interest stock of terms up to 10 years. Instead the matching portfolio based on 5 year steps contains 512 nominal of the 23 year stock only.

6.18. On further consideration however this feature can be understood. As stated in § 6.10 the liabilities to members approaching retirement begin to take on the monetary characteristics which are best matched by fixed interest stocks. A matching portfolio for such liabilities in isolation, i.e. disregarding the current pension payments, comprises the shorter term of equity assets and the longer term of fixed interest assets. A longer dated fixed interest stock was selected for the aggregate liabilities and the nominal holding is more than sufficient to cover the liabilities to pay current pensions. It replaces the shorter dated stocks that would have been selected to match the current pension liabilities in isolation.

6.19. Alternative calculations using different parameters for the standard deviation of interest and inflation rates, and involving different stochastic models, have been found to produce similar structures of matching portfolio.

Future contributions

6.20. The foregoing example of matching calculations for a prototype pension fund model allowed for future inflation of liabilities to members in service but ignored any future contributions. If future contributions are brought into the picture, the pattern of future cash flows is altered considerably and we are then concerned with matching the net liabilities of the pension scheme.

6.21. For the next examples the pension liabilities are as before and normal future contributions are payable until retirement at a fixed percentage of pensionable earnings. The current annual earnings of members will be taken as the same at each age, and the current annual rate of contribution in respect of members within each quinquennial age group will be taken as either (a) 20 or (b) 35. The matching portfolios for the net liabilities corresponding to (a) and (b) are as follows.

Prototype model F—with future contributions

Assets	Redemption dates	Nominal holdings (a)	(b)
Equity index	1– 5	0	0
	6–10	21	0
	11–15	200	143
	16–20	382	354
	21–25	142	80
	26–30	0	0
Fixed interest	23	308	0
M.V.		1,044	576
Matching rate (%)		9·8	9·4

These statistics may be compared with the first calculations ignoring future contributions which, based on 5 year steps, yielded M.V. 1706 and matching rate 9·9%.

Interpretation of results

6.22. The picture which emerges from these initial results is that a portfolio which consists in part of fixed interest stocks is appropriate for matching gross liabilities which are partly of the monetary kind, and that a similar type of portfolio can be appropriate for matching liabilities net of future contributions. However the effect of substantial future contributions can be to diminish the fixed interest content of the matching portfolio. In case (b) the immediate cash flow position of the fund is a positive income even before addition of investment income, and the matching portfolio is all equity.

6.23. It is also noticeable that the matching rate of interest diminishes with increasing future contributions. To understand this feature, recall the following aspects of the model:

Mean future rate of interest (%)	9
Mean future rate of dividend growth (%)	4·5
Dividend yield (%)	4·8
Fixed interest redemption yield (%)	11
Mean redemption yield on equities (%)	9·5*

* (since $1{\cdot}045 \times 1{\cdot}048 = 1{\cdot}095$)

It is probably fair to say that the matching rate of interest is a weighted average of the three rates of 9% on new money, 11% on fixed interest stocks and 9·5% on equities. It exceeds the 9% rate in all three cases because of the higher returns expected from existing investments. It reduces towards 9% when the requirement for fixed interest stocks is diminished and there is a significant future income to the fund which must be reinvested.

6.24. Further calculations which have been made indicate that the fixed interest component of the matching portfolio also vanishes when there is a substantial weight of projected benefits and contributions in respect of members at the younger ages. However the significance of this feature may well depend on the scale of the withdrawal decrements, a matter which has not yet been investigated. What can be said in general is that the matching rate of interest for any pension fund can depend upon the relative maturity of its current pension liabilities in relation to the net liabilities to active members, as well as all the financial parameters in the actuarial model.

Valuation of liabilities

6.25. Paragraphs 5.22 to 5.25 introduced the notion of matching as a technique of valuation to arrange any specified probability of ultimate surplus on the given assumptions. In the context of pension funds, especially those which are financed by the employer meeting the balance of cost, it seems natural to consider the valuation of liabilities by M.V. If this valuation of the liabilities is in balance with the market value of the fund and if the trustees of the fund invest in the matching portfolio, the consequence will be an approximately equal chance of ultimate surplus or deficiency, the extent of either being as immune as possible to future changes in economic conditions.

6.26. If this method of valuation were followed at the periodic actuarial investigations, the result ought to be a relatively stable funding rate which should not lead to significant under- or over-funding. However this desirable outcome would, as ever, depend upon the successful choice of a suitable actuarial model. The judgement of the actuary would thus remain paramount. In any case, apart from the theoretical considerations there are always practical aspects which may, within limits, be reflected in the presentation of an actuarial investigation.

6.27. Valuation by matching could be used to provide a particularly stable frame of reference from one valuation to the next—one which allows automatically for changes over time in the characteristics of the net liabilities of the scheme. In practical work there may be room for the subjective element, but it can be no loss to the actuary if he has at his disposal a more objective means of monitoring the position of a scheme.

Relationship with other valuation methods

6.28. Colbran[(12)] discussed the valuation of ongoing final salary pension schemes. He criticized the method of valuing assets by discounting expected cash flows, particularly when the fixed interest investments are discounted at a rate of interest which is below their current redemption yields. To the extent that valuation by matching involves a comparison between the M.V. of the liabilities and the market value of the assets, this approach accords with Colbran's views. However, it is meaningless to discuss the actuarial valuation of assets in a vacuum, without bringing into account the nature of the liabilities and the value placed upon them. As in so many actuarial problems, consideration must be given to the matching position.

6.29. We shall therefore now look at various alternative methods of valuation of final salary pension schemes, comparing each with the matching valuation. The following methods will be considered.

(1) Discount liabilities at the matching rate of interest and compare with the market value of the fund.
(2) Discount liabilities at a market rate of interest and compare with the market value of the fund.
(3) Discount liabilities at an average expected future rate of interest and compare with the market value of the fund.
(4) Discount liabilities at an average expected future rate of interest. Discount assets at the same rate after first assuming notional reinvestment at current market prices into a model portfolio such as two-thirds equities, one-third fixed interest stocks.
(5) As (4) but using the matching portfolio for the asset model.

6.30. Other methods could be added to the list, such as valuation of assets at recent average market prices, or by discounting the expected future proceeds from the actual portfolio. However the outcome of such methods will depend on factors which are not central to the discussion, such as vagaries in the stock market or in the portfolio of the fund from time to time. They can be thought of as sub-methods of the above classification.

6.31. Confining attention to the five methods listed above, the results of the alternative valuations of our prototype model fund with future contributions on basis (a) are as shown in the following table. Without any prejudice to the comparisons, the market value of the fund has been taken as fortuitously equal to M.V., namely 1044.

Valuation method:	(1)	(2)	(3)	(4)	(5)
Liabilities (L)	1,044	1,026	1,150	1,150	1,150
Assets (A)	1,044	1,044	1,044	1,165	1,149
L-A	0	(−) 18	(+) 106	(−) 15	(+) 1

Method 1

6.32. By its definition this is the matching valuation. In this example the matching rate of interest is calculated to be 9·8% (see §6.21). The liabilities are discounted at this rate but are still projected on the assumption of a mean rate of $7\frac{1}{2}\%$ annual increase in earnings.

Method 2

6.33. For method 2 the liabilities have been valued at a market rate of interest which has been taken as the 11% redemption yield on fixed interest stocks. For consistency the assumed average future rate of earnings growth has been increased from $7\frac{1}{2}\%$ to $9\frac{1}{2}\%$ p.a., but in accordance with the funding policy the allowance for pension increases remains at the fixed 3%. The latter aspect is why method 2 produces a more favourable result, to the value of 18, than the matching method 1. This more favourable result might be considered realistic on the grounds that current redemption yields on secure fixed interest stocks are the best available guide to future economic conditions. Whether this is so is a somewhat philosophical point; it is certainly not generally accepted.

Method 3

6.34. For method 3 the liabilities have been valued at 9% p.a. The result is unfavourable compared with the others, and the reason is the higher redemption yield available on existing investments, especially fixed interest stocks, coupled with the monetary nature of the pensions when in payment. If we adhere to the philosophy that the state of the investment market on any day is not necessarily the best guide to future economic conditions, we may suppose for the sake of the present discussion that model F is a realistic model for the future. If so, and if contributions are paid to the fund to arrange a 100% funding level by reference to valuation method 3, the result would be to produce a likely ultimate surplus with present value 106. There would be an implicit and perhaps unwarranted margin in the funding.

Method 4

6.35. For method 4 the liabilities and the assets of the notional model fund have been valued at 9% p.a. In order to specify the model fund fully it was necessary to decide upon suitable asset terms. The calculations were made using the 23 year fixed interest stock and assuming that the equities would also be sold at the 23 year date on an unchanged dividend yield. The result is a little more favourable than that of the matching valuation. The unwarranted margin of method 3 has been dealt with by including fixed interest stock in the model fund, but the precise result must depend upon the proportions of the equity and fixed interest assets in the model and upon their terms. Different model funds produce different answers. The range of possibilities is narrowed by consideration of the nature and term of the liabilities—in effect by matching considerations—but there remain degrees of freedom in the choice of notional portfolio.

6.36. In §4.24 it was pointed out that matching under a stochastic model imposes the same number of constraints on a portfolio as there are degrees of freedom in its choice. It can therefore be said that if there are spare degrees of freedom in the choice of model fund for the purpose of valuing assets, then not all the matching constraints have been recognized.

6.37. Does inattention to some of the matching considerations imply that method 4 is in some sense inferior to the matching valuation? This is another philosophical question to which there is probably no clear answer. However certain points can be made. In the first place if the mean term of the fixed interest stocks valued in method 4 were significantly longer than the mean term of the monetary liabilities, then an over-valuation could result in our example. In effect a mis-matching adjustment would be called for.

6.38. Similarly, if the quantity of fixed interest investments exceeded that required to cover the monetary liabilities then, depending on other facets of the valuation, it might be unwise to take full credit for the discounted value of all the fixed interest investments because in part they would be backing liabilities which are linked to inflation. This is Colbran's point. The answer to this point again lies in giving proper recognition to the matching position.

6.39. The case for preferring the matching valuation to the slightly arbitrary nature of the model fund is that it deals automatically with any necessary mis-matching adjustments. The case for preferring the approach of method 4 might be that the specified model represents the norm as perceived by the trustees of the fund for the purpose of considering their investment decisions. Whether this could be a point in favour depends on whether the norm is a reasonable one. If it differs significantly from the matching portfolio, it might be better to disregard the trustees' norm in the actuarial valuation. The trustees may be assuming the continued admission of new entrants to the scheme which we have chosen to ignore. Using method 1 the consequences of any relative investment profits or losses which result from departures from the matching portfolio would be brought into account in actuarial valuations after they have materialized. Using method 4 the profits or losses are capitalized before they materialize—which of course they may not necessarily do.

Method 5

6.40. Finally consider method 5, which resembles method 4 but uses the matching portfolio for the asset model. Although the values of liabilities and assets differ from those of the matching valuation, the result is broadly equivalent to that of method 1. Although there is a slight theoretical difference between the two methods, it is to be expected that in general they will produce equivalent answers. The choice between them is largely a matter of presentation.

Pension increases

6.41. The discussion on pension funds so far has revolved around examples using model F, in which the allowance made for future pension increases is a fixed 3% p.a. irrespective of the rate of inflation. Many pension schemes are financed more generously than this with a view to supporting the cost of pension increases which may be awarded at the discretion of the employer and/or trustees from time to time. What is the matching portfolio in such circumstances?

6.42. To answer this question it is necessary to be more specific about the funding policy in relation to pension increases. Model F implies a rate of pension increase 3% p.a. below the mean expected rate of R.P.I. If a new model is adopted in which the annual rate of pension increase is always 3% less than the *actual* growth of R.P.I. each year, then the matching portfolio is found to consist entirely of equities with M.V. 1,080 (contribution basis (a)). This is greater than the corresponding M.V. of 1,044 for fixed pension increases. The difference is a measure of the cost associated with linking the pension increases to inflation, even though the mean expected rate of pension increase remains 3% p.a.

6.43. In practice few pension schemes are funded on this type of index-linked basis. The target is more likely to be intermediate, such as 60% of R.P.I. or mid-way between a fixed 3% and R.P.I. Typically the target would not be specified this precisely and would form part of the overall strength of the funding basis. If the target is specified precisely, the means are at hand with which to implement the specified funding policy through the actuarial valuations. The technique is to adopt a matching valuation in which the matching portfolio is interpolated between that required for fixed pension increases and that required for fully index-linked increases.

6.44. The problem of comparing the values of pensions which are guaranteed or expected to increase at different rates was discussed by Johnston(13) with reference to evidence given to the Scott Committee. He referred to the approach devised by Brealey and Hodges, which was based upon consideration of the rates of return on portfolios which are appropriate to the liabilities. Johnston pointed out that if 'norm' portfolios could be established for schemes with and without index-linking, the difference in yield between them would provide a measure of the difference between the pension values. This is precisely the nature of the technique described above, in which the required yields are the matching rates of interest in respect of the different types of pension liability.

Investment performance

6.45. Holbrook(14) discussed the investment performance of pension funds in the context of two principles of investment policy:

(i) The portfolio should be constructed with regard to the nature of the liabilities.
(ii) Subject to (i), the objective should be to maximize the rate of return by investments which involve an acceptable level of risk.

6.46. In his paper Holbrook pointed out the difficulty of measuring risk for this purpose. The early theories of portfolio selection, referred to in § 2.12, measure risk in terms of the standard deviation of the return from an investment. Fama(15) improved upon this measure by relating risk to the liabilities of the investor. If a risk-free asset can be identified which is appropriate to the nature of the liabilities, then the performance of a fund can be measured in terms of the difference between the actual return and the risk-free rate. Holbrook indicated that the appropriate risk-free asset for a fund with fixed pension liabilities would be a matched selection of fixed-interest dated government stocks; the corresponding investment for a fund with liabilities indexed to inflation would comprise a similar matched selection of index-linked stocks. The risk-free asset in relation to the given liabilities is therefore the same as the absolute match as it is understood in this paper.

6.47. Absolute matching is not generally attainable, but we have identified the next best thing—the matching portfolio. By its definition this is the minimum risk asset relative to the liabilities. In principle the matching portfolio could be calculated for any given pension scheme at any time and the return on the fund could be compared with the return on the minimum risk asset.

6.48. This method of measuring investment performance could not realistically be used for the large scale comparisons which are made of pension fund returns. However it might be of use in measuring the performance of a fund from one actuarial valuation to the next, especially if the valuations are based on the matching method. If the techniques of actuarial valuation and performance measurement were harmonized in this way it would be possible to analyse investment performance and valuation surplus consistently between returns on the minimum risk asset and profits or losses arising from deviations of the actual portfolio from the minimum risk position.

Investment policy

6.49. Such a measure of investment performance could also be helpful when formulating investment policy, especially when considered in conjunction with the corresponding measure of risk in a portfolio. For this purpose risk in relation to the liabilities would be measured in terms of the standard deviation of the ultimate surplus or its present value. If the state of the market at any time is such that the matching portfolio does not seem likely to yield the best possible return on investment, as may often be the case, then it would be proper to consider alternative portfolios for investment providing the greater risk attaching to them is acceptable.

6.50. Modern portfolio theory is based on the Markowitz concept of an 'efficient portfolio', the objective of which is to maximize return relative to risk without regard for the nature of the liabilities. In principle it would seem feasible to consider 'efficient matched portfolios' in which the Markowitz measure of risk is replaced by the liability-related measure. Whether this idea would be practicable and worthwhile has not yet been studied.

Discontinuance solvency

6.51. It seems fitting to end this discussion with some consideration of the financial position of a pension scheme in the event of winding up. Gilley[(16)] discussed the dissolution of a pension fund and the problems which face the actuary when such an event occurs. Francis[(17)] described a case history involving the measurement of solvency (and the interpretation of what that meant) in relation to a particular ongoing scheme. Little seems to have been written about the actuarial problem of measuring the level of solvency in the hypothetical event of discontinuance of contributions to a fund which is not actually in the process of winding up.

6.52. The importance of the discontinuance solvency level in the financing of a pension scheme is well recognized, but so also are the difficulties in quantifying the measure of solvency with precision. There can be ambiguities in the precise entitlements conferred on members by the winding up rule. If the test of solvency is based upon the cost of purchasing annuities from a life office, the answer will depend upon the quotations received relative to the market value of the scheme assets at a certain time. Quotations from life offices can be expected to depend upon factors extraneous to the particular circumstances of the pension fund such as the matching position of their own pension business funds and the degree of competition between offices. Some pension funds are so large that they exceed most individual life office funds!

6.53. Actuaries are frequently required to certify the discontinuance solvency of pension funds, and a bare statement of solvency can usually be provided when the scheme is well funded. However not all schemes are well funded, and the auditors of company and pension fund accounts may in future call for more detailed information from actuaries than has generally been provided in the past.

6.54. When considering the discontinuance solvency level of a pension fund the actuary will normally be looking at the strict entitlements which arise on discontinuance of contributions, which for most funds will represent immediate and deferred pensions which are fixed or subject to fixed rate increases. These monetary liabilities could be met either by purchase of annuity policies or by continuing to operate a closed fund. Whichever may be the case, an actuarial assessment would depend upon the matching position as regards the length of the liabilities and the balance between current and deferred pensions. A matching valuation by reference to the fixed interest investment market would seem well suited for this purpose.

6.55. The suggested technique is as follows. First a suitable actuarial model is chosen. One such as model C of § 5.4 would seem appropriate because of the cautious assumption regarding reinvestment rates. A reinvestment rate in the range of one half to two-thirds the current market rate of interest would probably be generally accepted as prudent but not excessively so. Having established the model, the future cash flows of the discontinued pension scheme are calculated. The matching portfolio and its current market value M.V. are then determined. If required, the value of M.V. can be adjusted by the method of § 5.24 to establish any required probability of ultimate surplus. If the solvency level turns out to be marginal, the probability of ultimate insolvency can be calculated in relation to the chosen model.

6.56. If the solvency level is low, the technique can also be applied to determining the extent to which the portfolio can safely depart from the matching position. The mean ultimate surplus and its standard deviation can be calculated for the actual portfolio. If the probability of ultimate insolvency is greater for the actual portfolio than for the matching portfolio, then it would be prudent to switch to a better matched position.

6.57. Sometimes a request is made for the potential surplus on discontinuance to be quantified in terms of the uniform rate of regular increase of the strict discontinuance liabilities which could be supported by the fund. This 'bonus' rate of increase is less sensitive than some other measures of solvency to the details of the calculation and it conveys useful information. If there were a widespread practice of calculating and quoting such a statistic for pension funds there would be evident advantages—especially in relation to the pension aspects of company takeovers and mergers.

6.58. Here again the matching technique of valuation would have a part to play, because the matching position of a discontinued fund could be altered appreciably by variations in the bonus rate of increase of the pensions valued. The soundest approach would require alternative matching valuations with different bonus rates, finding by iteration or interpolation that rate of increase which balances M.V. against the market value of the fund.

7. CONCLUSION

7.1. This paper has concentrated on a single theme: that of a concept of matching which encompasses both interest and inflation and which leads to well defined and ascertainable matching portfolios. From investigations carried out to date the concept appears robust and useful. Applications have been described largely in relation to pension schemes, but potential adaptions to life office work are apparent.

7.2. There seems no lack of subjects for further mathematical and actuarial research, such as effects of demographic factors, alternative stochastic models for interest and inflation, linearly dependent asset models and the introduction of utility functions into the matching requirements. On the more practical side, it is hoped that enough has been said already to promote discussion and encourage others to pursue the ideas.

7.3. Finally I wish to express my gratitude to Mr S. Benjamin for his steadfast encouragement, to M. James for obtaining all the figures and to my other colleagues who commented on drafts. I am indebted to D. J. Shirtliff for providing invaluable support—in particular he identified the way in which inflation can be covered by the theory. I hope that I have done reasonable justice to all their efforts and I accept full responsibility for any errors which may, despite all checks, remain.

REFERENCES

(1) HAYNES, A. T. & KIRTON, R. J. The financial structure of a life office. *T.F.A.* **21,** 141.
(2) REDINGTON, F. M. Review of the principles of life office valuations. *J.I.A.* **78,** 286.
(3) SHEDDEN, A. D. A practical approach to applying immunisation theory. *T.F.A.* **35,** 313.
(4) BOYLE, P.P. Immunisation under stochastic models of the term structure. *J.I.A.* **105,** 177.
(5) TILLEY, J.A. The matching of assets and liabilities. *T.S.A.* **XXXII,** 263.
(6) FELLOWS, D. E. Pension fund liabilities and asset matching. *J.I.A.* **108,** 211.
(7) MOORE, P.G. Mathematical models in portfolio selection. *J.I.A.* **98,** 103.
(8) WISE, A. J. A theoretical analysis of the matching of assets to liabilities. *J.I.A.* **111,** 375.
(9) MARKOWITZ, H. M. Portfolio selection. *Journal of Finance.* **VII, No. 1, 1952,** 77.
(10) WEGNER, P. A quadratic programming formulation of the portfolio selection model. *J.S.S.* **16,** 468.
(11) BENJAMIN, S. The theory of games and its application to rate of interest. *J.I.A.* **85,** 373.
(12) COLBRAN, R. B. Valuation of final-salary pension schemes. *J.I.A.* **109,** 359.
(13) JOHNSTON, E. A. The comparative value of pensions. *J.I.A.* **109,** 1.
(14) HOLBROOK, J. P. Investment performance of pension funds. *J.I.A.* **104,** 15.
(15) FAMA, E. F. Risk and the evaluation of pension fund portfolio performance. (Supplement to the Bank Administration Institute Report).
(16) GILLEY, D. F. The dissolution of a pension fund. *J.I.A.* **98,** 179.
(17) FRANCIS, P. R. Solvency of a pension scheme—a case history. *J.I.A.* **106,** 319.

APPENDIX

Glossary of defined terms

	Main paragraph references	Definition
Basic assets	3.5, 4.25	Assets available for the construction of matching portfolios (q.v.)
Fully stochastic model	4.26	A stochastic model (q.v.) which is not determinate in respect of the interest and inflation parameters for any future time.
Invariant	4.14	A vector quantity with respect to which the scalar product with a cash flow vector is the same for any liability as for its matching portfolio (q.v.).
Matching portfolio	2.14–2.17	Generally, a portfolio which is matching in a specified way to the given liabilities. Unless stated to the contrary in the paper the positive match (q.v.) is intended.
Matching rate of interest	5.13	The rate at which the discounted value of the liabilities equals M.V. (q.v.).
Matching real rate of return	5.30	The equivalent of the matching rate of interest (q.v.) when the liabilities are discounted net of inflation.
Mean ultimate surplus (E_1).	2.14, 5.17	The mean surplus (or if negative, deficiency) resulting under a stochastic model (q.v.) when all the liabilities have been extinguished.
Mean square ultimate surplus (E_2)	2.17, 5.17	The mean square of the quantity resulting as described above for E_1.
M.V.	5.9	The market value on a specified date of the positive match (q.v.).
Positive match	2.17	A portfolio with no negative holdings which minimizes the mean square ultimate surplus (q.v.) in relation to the given liabilities.
Principle of absolute matching	4.2	The general property that if the liabilities can be matched absolutely then the matching portfolio must be the absolute match, irrespective of the stochastic model (q.v.).
Principle of invariance	4.14	The property, derived from the principle of absolute matching (q.v.) that the number of invariants (q.v.) equals the number of linearly independent basic assets (q.v.)
S.D.	5.17	The present value, discounted at the matching rate of interest (q.v.) of the standard deviation of ultimate surplus.
Stochastic model	2.10	An actuarial model in which rates of interest and (if relevant) inflation are uncertain but follow specified statistical patterns of behaviour.

ABSTRACT OF THE DISCUSSION

Mr G. M. Morrison (opening the discussion): The author's paper presents us with a model for the matching of assets to liabilities. Models are very familiar to actuaries; we have been using them in one form or another for over a century. Much of the work that actuaries do is involved in model building although I suspect that most of us do not realize it as such. Whether we are setting premium rates, valuing a life office, a pension fund or assessing general insurance reserves we are all using a model of one sort or another. Models are just a framework upon which we can exercise our actuarial judgement. We do not expect them to be borne out in practice and, indeed, we gain greater insight into our work by analysing the differences between the *actual* and *expected* outcome of an event.

I agree with the author when he says in § 1.1 that the concept of matching assets and liabilities is fundamental in matters of finance. There is a tendency, particularly with pension funds, to ignore the matched position of assets and liabilities, because we are told that it cannot be obtained. The trustees, investment manager and actuary have all too often worked in a vacuum because they have been unable to assess the investment strategy being followed relative to the liabilities that they are trying to meet. Against this background, whether or not it is desirable to match assets to liabilities is a separate issue. We should know what the matched position is.

The essential assumption of the paper is that the author is regarding the fund as if it were closed to new entrants. I think that this assumption is a little too strong and all that is necessary is that the fund has a finite life. Therefore, in principle, you could allow new entrants for a limited time to come into the fund. I am sure that the investment strategy and investment policy pursued by a number of pension funds would be completely different if we assumed that they were closed to new entrants. With this assumption of finite life, the ultimate surplus, as defined by the author in § 2.8 is based on a concept very familiar to actuaries. As soon as the stochastic assumptions are brought into play it is a natural and elegant extension to assume that the ultimate surplus just becomes a random variable. In the calculation of the ultimate surplus it is essential to bring in contributions and, indeed, in the case of a life office, the future bonuses that are to be declared. This would mean that the bonus rate would also have to be a random variable, but would be dependent upon the experience in earlier years. There is a bit of a conundrum here because future contribution rates and bonus rates for a life office are required which affect the ultimate surplus, which in turn may influence the contribution rate.

I now turn to the stochastic models which have been used by the author. Throughout the paper the author has assumed that lapse and withdrawal rates are deterministic. However, as he recognizes in § 2.10 it may be necessary to model withdrawals on a stochastic basis. I would go further; I think it is probably essential. Recently, many pension fund members have left the pension fund either by withdrawal, redundancy or early retirement. These withdrawals have probably far exceeded the deterministic rate and the assumption that the fund is closed will serve to shorten the liabilities, tend to change them from an inflationary to a monetary nature and hence alter the matching assets. In certain circumstances I would extend the stochastic model to mortality.

There are a number of observations which I would like to make about the financial aspects of the author's model. First, the author assumes that the rate of return follows an independently distributed log-normal distribution. There has been a considerable amount of research done by academics both in the United States of America and in the United Kingdom which tends to indicate that rates of return are log-normally distributed, particularly on equities. This research has been involved in testing the efficient market hypothesis. However, the Institute's Maturity Guarantees Working Party cast some doubt on the appropriateness of this model for the U.K. equity market in the long term. The Working Party felt that as far as U.K. equities are concerned, the market could be better modelled by working on the dividend yield and dividend growth rates rather than directly on returns. I think that any comprehensive model would have to distinguish between the gilt market and equity market more explicitly and, in theory at least, between U.K. and overseas securities. The gilt market is not just a single rate of interest. It is a whole complicated term structure of interest rates—and this should be recognized.

An important observation I would like to make in connexion with the variables in the model is one concerning the effects of lags, time delays and interdependence. In the model, historic inflation and hence inflationary expectations do not seem to influence the current rate of inflation.

My intuition suggests that a more sophisticated model which incorporates a lagged inflation effect would be more suitable. I believe in the market place that the inflationary expectations of investors are a contributory factor to both equity and bond prices. Therefore, I think that the equity and bond prices should have inflation as a variable. Finally, bond and equity prices are related in some way. Investment managers perpetually seem to be looking at the relative value of equities to bonds and I suspect this should be recognized in the model.

My comments on the stochastic models chosen are of some importance in themselves, but they do not question the validity of the author's approach. I can see no problems, apart from computing, in bringing in different stochastic models.

Finally, I would like to consider the practical implementation of the modelling technique proposed by the author. This is most interesting. Like the author I will ignore the application of the technique to life offices since I do not practise in this area. However, I do this in the hope that other speakers will rectify the omission.

Turning to the applications of the model to pension fund finance, it is reassuring to see that the author confirms the actuaries' basic feeling in matching assets to liabilities. In § 6.6 he confirms that if the liabilities are fixed in monetary terms, then the matching assets are solely fixed interest investments, and if they are entirely salary-linked then they are best matched by equities. I have mentioned earlier the effect of omitting withdrawals from the stochastic model. This is similar to the results shown in § 6.21 when it can be seen that the larger the proportion of the liabilities which is to be met by future contributions, the more the asset distribution is aimed towards equities. In a closed fund one would expect to see more fixed interest investments than equities.

The extension of the model into the valuation of the assets and liabilities is fascinating. With the exception of method 3 the results are all broadly the same. It would have been interesting if the excess of liabilities over assets had been expressed as an increase or decrease in the contribution rate. It is a complicated process to adjust the contribution rates to place the assets and liabilities in balance. An adjustment to the contribution rate will affect the net surplus each year, hence the matching portfolio and its M.V. Nevertheless, I suspect that the difference in the valuation method does not produce significantly different contribution rates under methods 2, 4 and 5. Caution needs to be exercised in interpreting these results. If you accept that the ideal position on the model is based on method 1, then to the extent that the assets held in the portfolio are not going to perform in line with that of the model due to a different asset allocation and say stock selection, it may be necessary to establish a mismatching reserve. I know of no pension funds which use the stochastic model for valuing assets and liabilities. They may do so in the future, as the author points out. Nevertheless, I see a major difficulty in presenting the results on this basis to the client. I dread to think what kind of response we would get from the typical pension fund trustee if we presented our valuation reports on this basis. Nevertheless, if this method is used in actuarial valuations I am sure that we must disclose what we are doing. We are entering a period of change in society's attitude towards pension funds. Disclosure is the order of the day. With the rôle of the actuary becoming more important and increasingly under scrutiny, it is a mistake to distinguish between the actuarial and disclosed valuation basis.

In §§ 6.41–6.44 the author shows how the methodology can help the actuary in estimating the cost of funding pension increases linked to inflation in some way. Any help in this area is extremely valuable. It could also be extended to the withdrawal benefit and early retirement problems which actuaries often face.

I am not sure that the matching portfolio is the minimum risk asset relative to the liabilities. It depends entirely upon how you define risk. In the discussion on P. G. Moore's paper (*J.I.A.* **98,** 103), Meyer Melnikoff, a visitor from the U.S.A., stated that "the concept of variability was not necessarily synonymous with risk and it would be more appropriate to distinguish between uncertainty, which was perhaps more directly related to volatility and variability, and risk, which might be better defined as the chance or probability of missing a target and by how much". Colin Lever, in the discussion on J. P. Holbrook's paper (*J.I.A.*, **104,** 15), also took this view. The author's definition of risk fits in with that of Melnikoff and Lever and is more helpful to the actuary over the long term, but I do not see how it will encompass the trustee who determines his risk relative to the possibility of an increase or decrease in future contributions at some future date. However, in some applications it is a far better measure than the variability of return which is increasingly used nowadays.

I am somewhat sceptical about why trustees get the performance of their pension funds measured. Perhaps the situation will change, but I am quite sure that trustees are not really interested in the risk that the fund has borne in the past. I doubt if they are even particularly interested in the risk that the typical fund will bear in the future. All the typical fund trustee wants to know is whether his rate of return is above or below average and what the return achieved is. The fact that the return may, whilst being above average, be negative or does not increase as fast as the liabilities is of no interest to him, certainly not in the short term.

The application of the model to investment policy is very interesting. The original work by H. M. Markowitz on portfolio selection was first published in 1952 (*Journal of Finance* VII, **1**, 77). It was discussed at this Institute by Moore, and his paper received a somewhat hostile reception. Markowitz based his work on a number of assumptions, including the acceptance of standard deviation as a measure of risk and the single-time horizon. A single-time horizon tends to make the investment policy more short term. This feature, combined with the definition of risk has led actuaries to be critical of the methodology. On the other hand, the author's matching process is long term and links the assets to the liabilities, thus making it appealing to actuaries. The author suggests in § 6.50 that 'efficient matched portfolios' should be constructed. The investment process is made up of a number of quite distinct judgements. First, one should set up a long-term investment strategy, and the model is very useful in this regard. Other investment decisions are probably of a more short-term nature, i.e. asset allocation in international equity markets, in industrial sectors and stock selection, where the shorter-term portfolio selection techniques may be helpful. In an ideal world one would blend the author's treatment with more short-term criteria, if you felt that you could out-perform the long term. This is clearly possible, perhaps by using the model as a utility function of the investors' preferences.

Professor A. D. Wilkie: I think that the author has been very ingenious, and made a considerable contribution to portfolio theory, perhaps without realizing it. Of course, he may have been more subtle than I am giving him credit for, in introducing portfolio theory to the actuarial profession, while pretending that he is not. Although in § 2.19 he contrasts his matching theory with the portfolio selection problem, in fact his matching model is an example of the portfolio selection model with various very helpful additions.

The portfolio selection model, as the opener has said, uses a single time-horizon. So does the author. His horizon is the time when all liabilities have been extinguished, and he is then interested in the ultimate surplus or 'terminal wealth' of the fund. His model is, therefore, appropriate to perhaps the shareholders' fund of a wholly non-profit life office, where the utility of terminal wealth for the shareholders is an appropriate function to consider. It is not really a multi-period model, where we may be concerned with the distribution of bonus over various generations of policyholders, or even the distribution of dividends over the years to shareholders. To bring in a multi-period utility function would complicate the problem in a way which neither I nor probably the author wishes to do at present.

In the portfolio selection model we consider a number of securities for which we know the multivariate distribution of terminal wealth, or at least the vector of expected values and the covariance matrix. The author restricts himself to having in the paper as he describes it apparently only one stochastic investment available at any one future time, defined by his various interest models. He creates different securities by taking each of his base assets, investing the proceeds at successive times in the one available investment thereafter and rolling forward the total to give the terminal wealth resulting from the base asset. Each base asset ei can be expressed as a linear function of the unit vectors uj, whose terminal wealth is the random variable $F(uj)$. We know the expected values and covariance matrix of these $F(uj)$s, so we can derive the expected values and covariance matrix of the $F(ei)$s, that is the terminal wealth from each base asset. The covariance matrix is given in his notation by ECE'.

The author treats liabilities in effect as a single negative asset, rolling forward the negative proceeds according to his investment model to give a negative terminal wealth, which is also a random variable. We know the expected value and variance of this random variable, and its covariance with the terminal wealth of each of the base assets.

We now have half the ingredients for a conventional portfolio selection problem. The other ingredients we would usually need are the present prices at which we can purchase the assets. This is an aspect which the author has ignored, and by so doing has simplified his model unnecessarily. He has chosen the portfolio that gives the minimum variance of terminal wealth, without considering the price at which he would have to buy the assets in that portfolio. He might find that, by choosing a rather different set of assets, he could get a portfolio with a somewhat higher variance of terminal wealth, but a considerably lower present price. This might well be a more desirable portfolio to hold. But, just as in the portfolio selection problem, there are now an infinite number of answers to the matching problem which lie along the boundary of efficient portfolios in the price/variance plane, that is, for any chosen variance of terminal wealth (higher than the minimum variance) there is some minimum price portfolio, and correspondingly for any price there is a minimum variance portfolio.

Alternatively, one can re-express the problem in the usual portfolio selection manner, assuming a fixed amount of money now, which may be invested in any portfolio of assets, and consider the return on this portfolio after meeting the stated liabilities. This return is just the residual terminal wealth or the author's ultimate surplus.

In the portfolio selection model there is also a unique minimum variance portfolio which can be found using the author's methods. The same constraints apply, that the covariance matrix must be invertible. The minimum variance portfolio is an important one. If a risk-free asset exists, it consists of that asset. The author has shown how to define and find the minimum variance portfolio when liabilities emerging over time are introduced, and there is not a set of assets that are 100% correlated with those liabilities.

The contributions the author has made are, therefore: first, you do not need to put a present price on future liabilities, but only need to specify the random variable which is the terminal wealth corresponding to these liabilities. Secondly, that an asset purchased now and switched at some future date into another asset can itself be treated as a further type of security. The fact that the number of available securities now seems to be infinite does not yet cause me concern.

Using these ideas we can generalize the author's model even further. Instead of having only one investment available for cash received at any date in the future we could have several. I have myself developed a stochastic model, in which the two forms of security considered are ordinary shares and irredeemable fixed interest stocks. It would not be too difficult to add a model for index-linked stocks too, since I consider inflation to be an explicit part of my model. We can now treat a £1 received at time t and invested in shares until time n as a different security from a £1 received at time t and invested in Consols until time n. From my model I can derive expected values, variances and covariances of these two securities, and of all securities with different times t. Also, since a particular fixed interest stock or, indeed, an ordinary share or an irredeemable stock produces a series of interest or dividend payments in successive years until it is either redeemed or sold, and since each of these money amounts can be invested in either shares or fixed interest stock for the remainder of the period until time n, we may end up with 2^n different possible securities for each basic asset. I readily admit that this may make the problem difficult to solve, but the question of finding a practical numerical solution is a different one from stating the correct theoretical solution.

Using this approach, we can also subdivide the liabilities into separate units, allowing their outcomes to be correlated with the outcomes of different investments, which is surely an essential way of approaching a with-profit portfolio, or even of looking at a pension fund where discretionary increases in pensions to compensate for inflation may or may not be given depending on the investment returns.

The solution to the problem I have just outlined will give us, not just a present matching portfolio, but a future investment strategy that is in some defined sense 'efficient', that is, it minimizes the variance of terminal wealth for a given expected value, or maximizes the expected value for a given variance.

We ought to consider other results from the theory of financial economics. If we were to set up a portfolio of assets which matched a portfolio of liabilities and was unbiased in the author's sense so that the expected value of terminal wealth was zero, but the variance was not zero, then that combination of portfolios of assets and liabilties should have a negative present value. You cannot expect someone to take on a risk unless the odds are biased in his favour. In effect, unless there were other considerations such as goodwill, you would need to pay someone to take over a fund which gave such a match. But it might well be possible to rearrange the assets at current market prices to give a fund with a positive expected terminal wealth, without increasing the variance too much, as I have already described. In that case one might be able to persuade someone to take over the whole fund. The value of such a fund has to be compared with the value of other types of assets available in the market, in effect with some version of the Capital Asset Pricing Model of portfolio theory.

I have referred several times to the market prices of the assets. In § 5.9 and onwards the author refers to 'market value'. I think you will see that often his market values are really notional values based on his models. Indeed, he says that the market value is independent of the actual assets held in the fund. By my definition the market price of a portfolio is directly dependent on the actual assets in that portfolio and their actual quoted prices in the market, and is independent of the liabilities and the actuarial model at that date. We have two different concepts and we need to give them different names.

Mr R. D. Masding: The differences between discounted asset value and market value approaches which are commonly used are largely presentational. The two should produce the same answer if the assumptions used are compatible. I do believe, however, that the 'natural' way of quantifying future liabilities is to identify and, therefore, evaluate the resources which will produce the cash flow to meet them. The author has, therefore, addressed himself directly to the fundamentals of the valuation problem. Having identified these assets I believe it is less confusing to the outside world if they are evaluated at their market values rather than in the currency of discounted values using a valuation rate of interest which differs significantly from market yields at the time.

Matching of assets and liabilities is clearly fundamental where resources are limited and security of benefits is at a premium. Thus the author's approach is directly applicable to life office work, to closed pension funds and, indeed, to the investigation of discontinuance solvency in any pension fund. However, when we turn to consider the investigation of the on-going financial position of pension schemes where the employer meets the balance of cost, we must not lose sight of the fact that the investment strategy which minimizes the volatility of the employer's future costs might also increase the expected level of those costs. The author refers to this point in §§ 6.49 and 6.50. Here it appears to me that the concept of risk should be related not only to the nature of the liabilities which are being funded, but also to the future resources which the employer can and is prepared to make available to the fund. Thus, at the end of the day, the investment strategy adopted might well be influenced as much by the nature of the employer's business as by the nature of the fund's liabilities and the extent to which they have already been financed.

Being a general practitioner in this area it is here that I find some difficulty in seeing how the techniques described in the paper will actually be applied. Let us suppose that the employer can tolerate a considerable degree of volatility in future pension costs, and an investment strategy and funding plan has been adopted which allows that the expected investment return will be higher than that on the matching portfolio. Moreover, the funding target, taking credit for this higher expected return, has already been achieved. It seems to me that the actuarial techniques employed in the valuation of the scheme must be sympathetic to the funding philosophy adopted, even though they may point only to the expected outcome. After all, we do have to arrive at a contribution rate which is appropriate to these circumstances. The question that the employer might well ask in those circumstances is, "given that I do not wish ever to see contributions above a specified level, what is the investment strategy which will minimize my contributions, subject to that constraint?" I hope one day we shall find an answer to that question.

Mr S. Benjamin: The one thing that I would want to know in reading somebody else's paper is how robust is the method, and he has really made an attempt to tell us how robust the method is and to give us a feel for it.

In § 5.20 the author says that the nature of the matching portfolio is relatively insensitive to the details of the interest model, and in § 6.13 he says that using 5-year time steps is often good enough, whilst in § 6.19 he says that different stochastic models of the market seem to make little difference, and different standard deviations of the interest and inflation rate make fairly little difference.

In § 6.44 he mentioned Brealey and Hodges. Now whether their model would actually be robust under the conditions here I do not know. The appendix to the Scott Committee's report had backing papers and those backing papers used a fairly complicated market structure with decision rules for market expectations inside each realization of the simulations they carried out, but I do not know how stable their results would be under variations of the parameter values, for example.

In spite of his best efforts—and they are very good indeed—to explain what he has done, I found that the example in § 6.11 is not quite as helpful as it might have been. That is the one with the simple pension fund and the matching portfolio produced for it. The author says he found the distribution of equity assets according to the various periods to realization broadly corresponds with the liabilities which are given in § 6.11. But in fact you will notice that there is no cash flow actually given. The liabilities are described and the assets are described, but it might have been more helpful to the reader at that point to show the emerging cash flow of each in order to substantiate the comment he makes.

I hope the author will forgive me if I compare his matching valuation with one that I tried myself many years ago and to which he refers in § 5.25. That was a matching valuation looked at by way of games theory. I did not use a probabilistic model; but it was not a simple deterministic model. Instead of using means and standard deviations I assumed future market rates of interest would lie anywhere inside a given range, an upper and lower bound. As a matter of supreme interest at the moment—and I am talking of 1959—I used 2%—and we knew that nothing would ever go below 2%—and 10%, because we knew that nothing would ever go above 10%. That was the model. Instead of taking a safety margin of two standard deviations, I looked at the worst that the market could do and the worst it could do to you by choosing low rates when you needed to invest or, strictly speaking, when you chose to invest, and high rates when you chose to disinvest. So, technically, it was a two-person zero sign game, and I found that my results were fairly stable too. In fact in both models the probabilistic model which the author has used and the games model, the aim really is to get as close to absolute matching as possible, and any deviation from absolute matching is actually penalized by the time intervals between the assets and the liabilities contained in a mismatching situation. I should like to make the point that I think that is a much better approach than taking a margin of the rate of interest as is laid down in the current valuation regulations, for example.

In pursuing this one by way of a small example given in § 5.19, which is the 5-year deferred annuity with just a group of assets available, the author gives the matching portfolio for that deferred annuity. If you accumulate the cash flow at, say, 7% and disinvest at 11%, that is 9% mean with 2% margin either way, and just use that simple approach, you will find that you are only trivially short of being able to pay for your last liability. The matching portfolio that the author produces does work on that model as well. You have about ·98 to pay for the last liability of 1. If, by contrast, just to see what happens, you take a completely mismatched portfolio, just a single deposit, and use 7% to carry forward, you actually require a deposit of 2·92 against the market value that the author gives of 2·45. So mismatching, by way of a single deposit, is fairly heavily penalized under both approaches, and that is because of the time intervals.

The author has developed his model and uses it much further than I was able to do. I have a suspicion that the games model could incorporate inflation. I have not tried. It certainly could incorporate irredeemables. They get discounted at the very last time period. What I did not succeed in doing and what the author has done is to keep dividends attached to their redemption capital. So I had a very simplistic model of the asset cash flow as just cash flow, and I could not actually deal with investments as dividends and capital. I could, however, measure the effect of a given portfolio, I do have a procedure to select one, but I could measure the effect of it, and in fact what I would do was to select by eye and calculate a multiplier which would make the assets cover the liabilities. In practice that might give an asset portfolio which might be fairly similar to the matching portfolio which the author puts forward.

Both models draw attention to the fact that there is no such thing as a value of liabilities and a value of assets in isolation, and they both actually point to a measure of profitability. There are several ways of looking at this, some of which have been mentioned; my approach is that if you think you can improve your yield by mismatching, then the increased yield ought to be measured and tested on the extra capital which is required to cover a mismatching position. If you mismatch you require more capital, and now consider whether the extra yield is worth it. The result may or may not be better.

Mr D. I. W. Reynolds (in a written contribution which was read to the meeting): The future usage by other actuaries of the matching process proposed may initially be directed to the pension fund examples discussed by the author but my belief is that they also will be of significance in general insurance. Here with the inter-relation of surplus and liabilities which may be a combination of inflation linked and fixed money values the author's ideas can produce a solution not readily available otherwise.

The final sentence of § 5.23 is perhaps the most significant in the paper. The significance lies in the fact that one can now estimate what is 'a cautious view of interest rates' through the effect it has on the matching rate of interest. Many readers of the paper will have had to take views on future interest rates which need to be cautious. I am sure they have not been able to relate the degree of caution to a single future valuation rate of interest. This is now possible using the author's method.

Mr D. J. Shirtliff: The essential idea presented in this paper is appealing in its simplicity and common sense. The author suggests that, of all the possible investment portfolios which a fund could hold, we should single out for special attention that one which minimizes the variability of the ultimate surplus or deficiency. He has then developed this simple idea into a mathematical technique which can be applied to the solution of practical actuarial problems.

If we wish to follow the author's lead by adopting such methods in our own work, we shall have to reconsider the valuation basis. Most of us have fairly firm views on suitable average values for future rates of interest and inflation, but perhaps little idea of their variability and, for many of us, no idea at all of the relative merits of different stochastic models. It is, therefore, reassuring to read that the author's final answers tend to be insensitive to the fine details of the actuarial model used. The reason for this is fairly easy to see, and has been referred to by Mr Benjamin. The method aims to find an absolute match if one exists, or the closest approximation if not. The effect of introducing random fluctuations into the valuation parameters is to penalize any mismatching by increasing the value of E_2. This leads me to a paradoxical observation that perhaps we can improve the accuracy of our calculations by exaggerating our lack of knowledge of the future. For instance, if the simple example in Section 3 is reworked with rates of interest of, say, 6% and 12%, we find essentially the same portfolio is chosen but the sensitivity to rounding errors is reduced.

The author has suggested that further research is required into linearly dependent asset models. In § 4.25 he states that assets which are dependent on other assets in the model should be excluded, because their cash flows can be generated by a combination of other assets. Whilst this is true for the unconstrained match, it is not in general true for a positive match. Furthermore, the exclusion criterion pays no regard to redemption yields. By arbitrarily excluding one asset or another from the model we may end up with a portfolio which is not, in fact, optimal. I do not know what the best answer to this problem is. It may be that we should include a full range of assets in our model, and then seek to minimize market value subject to E_2 being the minimum.

Alternatively, we could define a utility function of M.V. and E_2 and maximize that. Mathematically these are more difficult problems than the one which the author has solved for us, but their solutions would bring us closer to the conventional wisdom that a fund manager should aim to maximize return, subject to an acceptable level of risk.

Mr D. E. Fellows: In §§ 6.14–6.19 the author comments on matching by term. At first glance his conclusions appear to be different in certain respects from those which I expressed in 1981; but on closer inspection I think there is much common ground.

My approach was that the value of an inflation-proofed liability would be unaltered on a change in the rate of interest if the real rate of return, that is to say the difference between the interest rate and the inflation rate, remained unaltered; but that in the case of an existing fund the value of assets, other than cash on deposit, would be affected by an interest rate change. The degree of the change in asset values is sensitive to the outstanding term, contrary to the author's suggestion in § 6.15.

It is when we come to consider the position of equities that the reason for the different conclusions becomes apparent. The author acknowledges, in § 6.16, that his model assumes that equities can be sold at future dates on market terms which are independent of future interest rates. This is a reasonable assumption in the sense that a change in interest rates should, at least in part, be counterbalanced by corresponding changes in inflation and dividend growth prospects and hence make market values somewhat insensitive to interest rate changes. In practice, however, this is not necessarily the case. As the author suggests, we need to substitute index-linked gilts for equities to make his approach valid, assuming that revaluation is on a prices basis throughout, rather than on earnings. The apparent inconsistency between the conclusions in the two papers can, therefore, be explained by the differences in the models and by the introduction, later in 1981, of index-linked gilts.

It is also important to recognize that the matching implications for inflation-proofed liabilities are affected by the degree of correlation between the real rate of return and interest rates. For example, if the real rate of return falls as interest rates and the rate of inflation rise, the value of such liabilities will increase and the value of most types of long-term asset will reduce, thereby aggravating the solvency position of the fund. It could, of course, be the other way round. Matching by term for pension funds could be specially sensitive to the pattern of behaviour between real rates of return and interest rates.

Mr E. M. L. Beale (a visitor): I have just two technical comments on this very interesting paper.

I was surprised at the assumption that the matrix $\mathbf{E}\,\mathbf{D}'$ can be inverted. This is because I would have expected that the number of rows in $\mathbf{E}$, which is essentially the number of possible investments, would in fact exceed the number of columns, which is the number of time periods at which you have to make your matching. If this assumption is relaxed I do not think this makes any fundamental change to the model. One still has a useful model which can perhaps be best expressed in conventional portfolio analysis terms as requiring to minimize the variance of the ultimate surplus for a given mean ultimate surplus, assuming a fixed initial value for the portfolio. Some solution in this one-dimensional family of solutions will then minimize the probability of an ultimate deficiency.

My other comment concerns Mr Wise's statement in Section 3 that the computing is very sensitive to rounding errors. This is because the matric $\mathbf{C}$ is nearly singular. To analyse this problem, define $\mathbf{h}$ as a row vector with j^{th} component equal to the mean ultimate surplus produced by one unit of money becoming available at time j. Then the mean surplus from the portfolio, say m, is defined by

$$m = (\mathbf{xE}-\mathbf{l})\mathbf{h}'.$$

Now if there were no variability in the returns we would have

$$\mathbf{C} = \mathbf{h}'\mathbf{h},$$

and if we write

$$\mathbf{C} = \mathbf{h}'\mathbf{h}+\mathbf{R},$$

we find that E_2 reduces to m^2+v, where v is the statistical variance of the ultimate surplus, defined by

$$v = (\mathbf{xE}-\mathbf{l})\mathbf{R}(\mathbf{xE}-\mathbf{l})'.$$

This is the same form as in § 3.8, but with $\mathbf{R}$ replacing $\mathbf{C}$. Since $\mathbf{R}$ is much better conditioned than $\mathbf{C}$, it will be computationally advantageous to use this formula. This means introducing m as an intermediate variable, and either introducing a Lagrange multiplier on the equation defining m in terms of the x_j, or substituting for one of the x_j in terms of the others and m.

As Mr Wise observes, there are many algorithms for quadratic programming. But if the number of possible investments is large, then it would probably be better to keep the net receipts at each time period as intermediate free variables, so that the quadratic part of the objective function would only have as many rows and columns as there are time periods, and to treat the problem as a general quadratic programming problem.

These are technical comments which do not invalidate or attack the substance of Mr Wise's paper in any way.

Mr P. N. Thornton: There are three areas in particular where the techniques which the author has developed will be of great practical value in our work for pension funds.

First they reconcile the market valuers and income discounters. The reason they do so is because on the chosen assumptions about future investment and inflation conditions, the matching portfolio is necessarily determined in a way which involves the actuarial assumptions used for projecting the liabilities.

Whichever method of selecting the results is adopted, the assets and liabilities may be said to be valued consistently. It is here where the income discounters have felt that there is what you might call a 'consistency gap' between liabilities valued by discounting future projected cash flows of benefit outgoings to present values and assets taken at a market value which may, but may not, imply consistent future rates of investment return. What the paper implies for market valuers is a market value of liabilities which may well be different from what they would have calculated before, and hence lead to different valuation results.

This reconciliation will be particularly helpful when valuing pension funds which are closed to future entrants but otherwise continue in full force for the present members. At the point of initial closure one would naturally tend to use the normal on-going valuation method, while eventually, when all members have retired on pension one would be using a market value of assets discounting the liabilities at market rates of interest. In the past the transition between these two different styles of valuation basis has been very much a matter of judgement. The author's paper will in future assist greatly in this area, as valuations involving his matching portfolio will provide a bridge from one type of basis to the other.

A related area of interest is that of discontinuance solvency. Some actuaries tend to regard this as an objective measure, but I do not. In practice when a scheme is wound up and quotations for immediate and deferred annuities are obtained from a number of insurance companies, the variation in costs quoted is typically up to one-third and sometimes more. In the course of negotiation, the forces of competition result in some insurance companies offering significant reductions as they whittle down their reinvestment and mortality margins. For this reason many pension scheme actuaries have preferred to assess the cost of buying out the winding-up liabilities on their own assumptions. The author's method will now provide a significant advance in the objectivity of such assessments, and should provide a basis on which insurance companies can provide realistic larger-scale winding-up quotations.

This leads on to the question of funding levels. By applying the technique to investigate the winding-up position on the basis of various future rates of benefits increase to be provided for, we shall be able to judge the level of solvency at the valuation date in a way which reflects current market conditions more directly than might otherwise be the case with our chosen method of valuation for the scheme on an ongoing basis. One possible use of the technique which I think should be of interest to pension fund trustees, and to auditors, relates to what we call the 'past service liabilities on an ongoing basis'. This measure tends to be very dependent on the particular method and assumptions adopted. Usually allowance is made for future salary increases, but these only arise out of future service. Instead 'past service liabilities' could be defined in terms of the benefits which we would wish to provide on a winding up of the scheme. For example, these might be deferred pensions based on current salary with provision for future increases in line with price inflation rather than salary inflation. We have tended to use a measure of 'past service liability' which involves the projection of future salary increases as a consequence of our funding methods, but a statement of the index-linked winding-up position may be of more relevance for solvency statement purposes. The author's technique will provide an objective method of assessing this.

Mr A. F. Wilson: The science of the actuary is based on mathematics. It has always concerned me how few of the techniques we employ in traditional actuarial areas use mathematics developed in this century. I have little doubt that many actuaries will view the complicated mathematics in this paper with dismay. Yet if the profession is to progress and flourish in the next century it is important that we refine our methods in the light of the mathematics of the twentieth century.

If there is a desire to be cautious, for example, in a life office valuation, a statistician would make the best estimate of each of the variables in the underlying distribution, and then add suitable multiples of the standard deviation to the mean to achieve the desired result. The actuarial approach hitherto has been to bias each variable on the cautious side, often by an arbitrary amount, and then assume that the expected value derived from the resulting distribution gives the required degree of certainty. The problem is that you have no way of knowing whether the answer is reasonable. This 'actuarial' approach is inculcated into new actuaries through the course of reading, and appears to lead to an unwillingness to take realistic estimates of variables. The introduction of techniques which will help to break this mould are to be welcomed.

I want to concentrate on the implications of the theory for pension funds. The author has shown that if allowance is made for some linking of pension increases to inflation as well as for future service within the fund, there is little need to hold any fixed-interest securities in a matching portfolio before taking into account any future entrants. If one takes into account new entrants for a sufficient period, which may be as short as the period to the next valuation date, the need to hold any fixed-interest securities disappears. I believe that it is right to make some allowance for new entrants. Curiously, for exactly the reason that the author says in § 2.7 that they should be excluded. Unless the valuation basis used is as realistic an estimate of the future as it is possible to make, the valuation is biased towards future participants in the fund if the basis is strong or to present participants if the basis is weak. Allowing for new entrants reduces the cross-subsidy between generations rather than the reverse. It is also closer to the truth.

I believe that more actuaries are realizing this as the effects of the recession on the fortunes of pension funds become analysed. In consequence, I believe that the direct application of this paper to future valuations of the ordinary pension scheme on an ongoing basis leads automatically to a valuation made in accordance with § 6.29 method (4), but assuming that the model portfolio is entirely of real assets. The arithmetic involved in applying the full force of the matching technique method is formidable, and the differences in the valuation result are likely to be too small to warrant the extra work involved in applying the author's method in full.

Matching will be much more important when dealing with a fund which is already closed to new entrants or which has suffered a considerable decline in membership. However, even for such a scheme there will be many benefits which are directly linked to inflation, and the majority of assets may still need to be real assets. In such circumstances there remains the question of whether the fund should invest in index-linked gilt stocks or in equities. (For simplicity I have ignored property.) Let us, for example, consider the figures in § 6.12 for model F. Under the 5-year model these show an M.V. of 1706 with a standard deviation of 3·1. To be virtually certain of meeting the liabilities matched assets with market value of 1715 would suffice, that is, three standard deviations more than M.V.

The model for equities which leads to this conclusion is in fact much more akin to assuming that the fund is invested in index-linked gilts, since the total return is linked directly to the rate of inflation, and exceeds it by about $3\frac{1}{2}\%$. Suppose we took this as the index-linked solution, but were to consider as an alternative portfolio one with real assets entirely in equities, which were expected to yield a real return of 5%, but subject to variation year by year either because dividend growth fluctuated or because the yield on which the equities were bought and sold varied. The M.V. for such a portfolio for model F might be of the order of 1500 whilst the standard deviation might be as much, say, as 80. What strategy should be taken in such a scenario if the actual assets had a market value of 1600, 1700 or 1800? If the value were 1800 the benefits would effectively be certain under either investment portfolio. Yet the expected profit if index-linked is chosen would be only 94 whilst if equities were chosen the expected profit would be 300. It would clearly be of advantage in such a circumstance to invest in equities. The position if the value of the assets were only 1600 is more of a problem. There is only about a 30% probability that the liabilities would be met by investing in equities. There is no chance that they will be met by investing in index-linked stocks. Which is now the better strategy? That may well depend on other circumstances, such as the employer's ability to pay. One thing is very clear, the techniques developed by the author now enable us to pose the right question in terms that can lead to a meaningful discussion and decisions.

I believe that this ability to compare and contrast the attributes for different portfolios is possibly more important than the strict matching theory. We now have the basic tools to address ourselves to the important question of what extra expected yield we require from equities to cover the risk of greater variability of return.

Mr R. B. Colbran: The author makes substantial claims for his paper. He claims he has given us a new study and a new theory which lead to specific portfolio structures, some particularly applicable to pensions. Nevertheless, the mathematics is sufficiently intricate that it has to be put in a separate paper, and even the key expression 'fully stochastic' is only defined in the paper by telling us what it is not. Common sense suggests that any structure of mathematics built upon a model can only be as sound as the underlying model. Looking backwards over investment history this century or perhaps just the last 40 years I find it difficult to envisage a probability distribution that one could fit to the actual rates of interest and to rates of inflation as standard deviations, let alone to look forward and suggest that I could postulate an average rate of interest and a standard deviation. Nevertheless, the author does claim, and Mr Shirtliff supported this, that the actual choice of the model makes very little difference to the results. I find this surprising, and I feel that we are being asked to take a great deal on trust at this stage, and we should need more convincing before we could accept the author's ideas as a fundamental change in our approach.

It did make me wonder whether a more limited approach was required. For example, to actually construct a number of models and derive matching portfolios for each of them, when the differences might give us some estimate of the risk. That would fit well perhaps with a portfolio of index-linked annuities backed by indexed stocks. Again, with a final salary pension scheme there seem to be so many variables and unknowns that one wonders at the realism of the method. However, the author said he has made a calculation on an actual fund. One would think that the arithmetic would be at the scientific order of computing, but perhaps that is not so.

In the paper, I have a personal mention in connection with the discounted method of valuation of assets. Mr Thornton suggests that the author has actually reconciled the market value proponents and the discounted value supporters. The discussion in this context has to be in the context of final salary liabilities. When the author deals with this in § 6.38 he seems to put substantial emphasis on the money part of the liabilities. Clearly when people are on pension and the liability is fixed in money terms, we are talking about a different approach. But where we are looking some distance away, it seems to me that the discounted value approach at least in the context of fixed interest assets, depends on their being an absolute money rate of interest. I thought that when we value a final salary scheme we look at a real return, a return in terms of investment yield in excess of average earnings, and if we are looking at that, there is not an actual money rate of interest involved in the calculation until we reach the point of retirement. Yet in the author's illustrations in § 6.28–6.38 he refers in all his examples to an absolute rate of interest, and this is one point where I feel there is a fundamental difference between the market value and the discounted value approach. With discounted values it is claimed that there is a money rate of interest which is of importance in this calculation.

While I do not, of course, want to discourage research and new thought, I suggest that there is some danger of trying to pretend that the valuation process of a final salary scheme is more scientific than it really is, and we should be realistic in our ambitions for that process.

Mr P. E. B. Ford: An item which was presented as a major problem in Redington's paper was financial options in the liabilities: for example, guaranteed surrender values or open market options on self-employed annuities. When future research is done based on this stochastic approach if we are going to have a practical set of results, this particular aspect will need to be tackled and solved, because increasingly we do give these financial options, and if we do not come up with a solution, the model we use will not be sufficiently robust to produce practical solutions to life office problems.

Mr T. Grimes: I should like to make two points. One is practical and the other is slightly theoretical. This paper brings mathematics into an area where we have previously, some of us in the investment field, pretended we had some actuarial knowledge. This paper gives us an idea, not of the way in which the funds actually have to be invested from minute to minute, but a way which should give us a chance to fix the basic portfolio, that is to say, the portfolio against which the actual investment manager will be measuring. This is something that has, in the past, only really been done by rule of thumb.

A pension fund which has index-linked liabilities has presumably been advised by its actuary that it should invest in equities (now index-linked stocks) more than in gilt-edged stocks. Likewise with a closed fund the actuary will have probably suggested more gilts, but how many more we have never really been able to say precisely. This paper will enable us to calculate that a little more accurately but not precisely. There are still going to be market situations which will give the investment manager opportunities to do a little better than the basic or calculated portfolio, and we have to give the investment manager that freedom.

I see the negotiations between the actuary and the pension fund trustees in future including a recommendation for a basic portfolio in which the fund should be invested. But not only that, also a suggestion that if the fund is invested in any different way, the actuarial valuation ought to be adjusted to allow for the additional risks which will come from that portfolio investment. Clearly the greater risks will be expected to produce appropriately greater returns.

The theoretical point I want to make is concerned with immunization. If immunization were possible it would give us a way of always making a profit. It would not matter which way the market moved. If your fund is immunized you will be making a profit. It does not matter what stochastic model you assume, or whether the market follows that stochastic model. It is too good to be true, of course. But in those circumstances the immunization technique actually gives the opportunity to bias the investment results positively. The techniques described in this paper are based on the concept that you minimize the sum of the squares of the deviation of the final result. This gives an equal chance of beating, or failing to beat, the eventual outcome, that is to say, your assets may be more or less than the liabilities at the end of any period. A conservative actuary would have preferred an investment model which was positively skewed, that is to say, which gave a greater possibility of profit than loss. The trouble is that the mathematics is a lot more complicated. It is not possible to use the techniques involving matrices, etc., of this paper. However, the exercise will be worth doing to see whether it makes any difference to the portfolios which arise from the technique. It is quite possible that, given the robustness of the technique, the answers will be the same or close enough for us to be able to use them.

Mr J. G. Spain: One of the pressing problems these days is in relation to the partial dissolution of the scheme when a business, whose participating employer has been participating in some group scheme, decides that the scheme will be discontinued. Actually giving a value to another actuary and saying this is the way I have valued these liabilities for this particular set of members, and the way I have done it is set out in *J.I.A.* will not cut a great deal of ice because some of the results are not going to be verifiable. If one goes back to the idea, say, of triennial valuations which reveals a surplus on particular bases, it has normally been considered advisable to do some investigations into how that surplus arose, i.e. a reconciliation of surplus. I would find it difficult to inform a trustee that the reason for the difference in results is not because I have changed my valuation rate of interest or whatever, nor because they have had lots of new entrants aged 15. It is because I have changed the lognormal distribution into something else which seemed more appropriate. I do not think there is a case for going for new mathematical procedures, solely for their own sake.

Mr K. G. Smith (closing the discussion): I like the opening quotation of this paper which made it clear that the aim was directed to understanding rather than to actually changing our practice or, as the author puts it, "refinement of the concepts". He has made an ambitious attempt to build a model which predicts the variations of equities, fixed interest and index-linked stocks, and then matches them to predicted variations in liabilities so as to produce a minimum variation.

The length of the bibliography indicates that matching and immunization are ideals which have fascinated actuaries for many years. The author's inclusion of an appendix with a glossary of defined terms was most useful to me. It could possibly even be expanded to cover items such as cash flow vector and scaler product.

Paragraph 2.7 which enunciates the closed fund concept is fundamental to the author's approach, but seems to me to limit considerably the relevance of his solutions. Proportionately few funds are closed to new entrants and, notwithstanding some reductions in membership, few are unable to meet current outgoings from current income. However, Mr Thornton pointed out that there are a number of closed funds to which the paper would be relevant, and Mr Spain mentioned relevance to partial dissolutions which occur fairly frequently.

It is clear from the later example in §§ 6.20–6.24 that future contributions of current members have the effect of lengthening the time horizons for realization of securities and thus of rendering irrelevant temporary variations in the market value in the meantime, provided interest and dividends are unaffected. *A fortiori* presumably future contributions of new entrants must further extend the time horizons leading to an all equity portfolio, a point made by the opener. Mr Wilson also stressed this point. However, the closed fund concept is one normally adopted for actuarial valuation, so perhaps it is helpful to appreciate its implications.

Like the opener and other speakers, I, too, found the author's idea of an ultimate surplus which varies with experience a very helpful one. His simple comparison of the respective characteristics of the objectives of immunization in portfolio selection and matching is useful, and emphasizes his search for a unique solution. I wish I could pretend I fully understood the theoretical analysis in Section 4 which follows the worked example. I imagine also I might not have been the only one who breathed a sigh of relief to find that the associated paper covering theoretical analysis had been published separately (presumably providing an alibi for not having studied it). It was, therefore, gratifying to have Mr Wilkie's and Mr Benjamin's contributions to this aspect of the paper.

However, when it comes to the examples in Section 5 I feel a shade more confident. This interesting section deals with the matching of a limited choice of index-linked stocks against index-linked annuities, a practical problem for those offices which now provide such an option out of their compulsory purchase pension annuity funds. Mr Benjamin drew attention to the robustness of the matching profile, and he also gave interesting comparisons of a model based on games theory which could be adapted to include inflation. Mr Masding drew attention to the advantages of the matching portfolio at market value.

When we come to the application to pension funds in § 6.2 the author assumes for his model a rate of inflation with a mean of 6% and a standard deviation of 1%. Using my pocket calculator I derived the figures for annual inflation over the last 12 years. These vary from 24·9 to 5·3%; they had a mean of 12% and a standard deviation of 5·6%. This differs somewhat from the author's average of 6% and a standard deviation of 1%. Have recent months possibly lulled us into a false sense of security or is the author an optimist? Mr Colbran drew attention to the difficulty of matching inflation to a model.

Paragraphs 6.6–6.10 summarize the most important if not entirely surprising conclusions, namely, that benefits increasing with salaries or inflation should be covered by ordinary shares, and monetary liabilities by fixed interest. An important point is made that even where pensions are indexed, either formally or informally, if guaranteed minimum pensions are excluded as is logical since they are indexed by the State, the GMPs become monetary liabilities.

Paragraph 6.24 makes the relevant point that the fixed interest component of the matching portfolio vanishes where there is a substantial weight of projected benefits and contributions in respect of members at the younger ages.

Mr Fellows discussed the apparent inconsistency between his 1981 assumptions and those of the author. Matching of inflation-linked liabilities was sensitive to the real returns on equities reflected in their prices on realization. The issue, subsequently to his paper, of index-linked gilts have provided an interesting alternative.

Mr Wilson drew interesting comparisons of index-linked stock and equities and outlined the possibility that a lower cost could be associated with a larger variation and still prove cheaper within the margin of anticipated variation as measured by the standard deviation.

In conclusion, as he emphasizes, the author has concentrated our minds by simplifying his models so that they consist of only two types of investment. In practice, investment managers have a much wider field to choose from—not only U.K. equities and fixed interest, but also U.K. property, overseas equities, overseas property, and index-linked gilts, not to mention the more esoteric fields like options or works of art. Furthermore, we should not forget that the aim of investment managers of pension funds, closely monitored by their trustees or the employer, is to maximize returns within constraints of permissible variation, and it would be a counsel of despair to them to assume that their skills either in predicting future market variations or in selecting individual securities are absolutely negligible. In other words, they would and do deliberately mismatch, for example, by buying overseas securities against U.K. liabilities if they are convinced of an average positive advantage,

The author made the interesting suggestion that the performance of these mismatches ought to be measured against the matched portfolio, and Mr Grimes also referred to this aspect.

The President (Mr C. S. S. Lyon) (proposing a vote of thanks): For the older ones among us this afternoon, I think discussion of a paper of this kind must cause us to cast our minds back 30 years or more to the paper which has been referred to on a number of occasions, namely, F. M. Redington's review of the principles of life office valuation (*J.I.A.* **78**, 286). It seemed to me that two or three sentences in that paper were a very useful indication of where the author's approach was to lead him. Mr Redington wrote at that time: "Whenever an office accepts a new contract or makes an investment, it affects the balance of the business as regards its sensitivity to interest changes. What is that effect? What is the effect desired?

The author has brought to us a definition of matching as a technique of valuation to arrange any specified probability of ultimate surplus on the basis of given assumptions, which seems to me to be a way of answering the question that Redington posed. The author went on to say that a direct measure of the degree of mismatching is the minimum value of the mean square ultimate surplus which is attained by the matching portfolio. The paper, therefore, marks a new point of departure in the design for tools quantifying the effect of mismatching, not just for pension funds but, as we have heard, also for life assurance and general insurance.

It is impossible to tell today what family of future papers it will generate, but of their generation there can be no doubt. I predict that in the not too distant future our students will be expected to master the mathematics, though I do agree with the speaker who thought we would not be able to communicate it to trustees.

Tonight's paper marks one further step along the road that my predecessor, Fred Menzler, urged us to follow when he proposed the vote of thanks to Mr Redington some 30 years ago. "I have felt", he said, "that as a profession we tend to be too much obsessed with present values. These all too convenient summarizations sweep everything up into a single portmanteau figure, but as is so often the case with financial-cum-statistical summarizations, that clear view of the wood may cause us to forget the trees or, in other words, the series of financial events in time for which we are called upon to make provision".

I am sure we are all most grateful to the author for taking us a step further along that road—along the road that Redington began to lead us—and I would ask you to accord him a warm vote of thanks in the usual way.

WRITTEN CONTRIBUTIONS

The author replied briefly at the meeting and subsequently wrote as follows:

Among the many interesting points made in the discussion I should like to respond to two in particular. They concern the assumption to be made concerning new entrants and the relationship with portfolio theory.

I accept the point made by several speakers that the assumption of no new entrants is unnecessarily restrictive and that the fundamental requirement is that of a finite lifetime for the liabilities. Therefore the theory can be applied with the assumption of new entrants for a limited period. The choice of period during which new entrants are to be hypothesized could be somewhat arbitrary, and I think the choice would best be governed by other aspects of the proposed application of the theory. For example, if the application were an actuarial valuation in which allowance is to be made for new entrants during a period of 10 years, it would be logical to carry out any type of valuation—whether by matching or by discounted cash flow—using the same basic assumption. If the application were investment strategy the new entrant assumption might differ from that of the actuarial valuation. I alluded to this in § 6.39.

The second point concerns portfolio theory, and I thank Professor Wilkie for his analysis of the key points of difference between the approach adopted in my paper and that of the conventional portfolio selection model. Having discussed this subject with him on occasions subsequent to the Sessional Meeting I should like to develop his points a little further. Portfolio theory is concerned with the mean return on a portfolio and the variance about the mean, but my approach is concerned with minimizing the mean square ultimate surplus and not its variance. At first sight this difference may not look very significant; indeed, the matching calculation could be re-worked by reference to the variance of ultimate surplus.

The worked example given in Section 3 provides a good illustration. In § 3.8 I referred to the covariance type matrix **C**. In order to minimize variance instead of the second order moment we must repeat the calculation after replacing **C** by the true covariance matrix, which is the residual to which Professor Beale referred. The minimum variance portfolio turns out to be precisely 10 units of the 3-year stock, and the variance of surplus which results is precisely zero. This is easily verified because the cash flows from the minimum variance portfolio net of liabilities are zero at times 1 and 2 and there is no uncertainty in the outcome of cash flows at the horizon time 3. However, we have now succeeded in miminizing variance at the expense of the mean! The mean ultimate surplus is 9, without variance, and such certainty of outcome can only be achieved in this particular example by use of a portfolio whose market value would be several times greater than that of the matching portfolio.

In the conventional portfolio problem the market value of the fund is fixed at any date, but in an actuarial problem generally there is no uniquely defined "market value" for the liabilities. In order to make sense of the minimum variance solution it therefore becomes necessary to replace the market value constraint by a constraint upon the mean surplus. The portfolio which has minimum variance out of all those which give rise to a mean ultimate surplus of zero is of course the same as the positive unbiased match of § 2.14. The simpler constraint of minimum mean square ultimate surplus produces results which seem almost indistinguishable from those of the positive unbiased match, but as Professor Beale pointed out there may be computational grounds for working with the latter.

Before leaving Professor Wilkie's contribution I would like to comment on his final remark concerning market values. While it is true that all market values quoted in the paper were constructed by valuing at specified rates of interest, that was really only a simplification for the purposes of presentation. In a realistic application, of the type which I have carried out since writing the paper, actual market values of quoted investments can be used. The way in which market values enter the calculation is simply in placing a market value on the matching portfolio on any specified date.

Finally, Mr Benjamin asked for a statement of the cash flows resulting from the liabilities and matching portfolio referred to in § 6.14. On the assumption of earnings increasing at $7\frac{1}{2}\%$ p.a., in accordance with the model, and using five yearly steps the cash flows are as follows:

Time	*Liabilities*	*Matching portfolio*
$2\frac{1}{2}$	612	621
$7\frac{1}{2}$	850	873
$12\frac{1}{2}$	1,151	1,194
$17\frac{1}{2}$	1,418	1,474
$22\frac{1}{2}$	1,222	1,201
$27\frac{1}{2}$	459	0

Once again may I express my thanks to the contributors for a most constructive discussion.